Nonlinear Schrödinger equation and N-soliton interactions: Generalized Karpman-Solov'ev approach and the complex Toda chain

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A method for the description of the N-soliton interaction, which generalizes in a natural way the Karpman-Solov'ev one for the nonlinear Schrödinger (NLS) equation, is proposed. Using it, we derive a nonlinear system of equations describing the dynamics of the parameters of N well separated solitons with nearly equal amplitudes and velocities. Next we study an exhaustive list of perturbations, relevant for nonlinear optics, which include linear and nonlinear dispersive and dissipative terms, effects of sliding filters, amplitude and phase modulation, etc. We prove that the linear perturbations affect each of the solitons separately, while the nonlinear ones also lead to additional interactive terms between neighboring solitons. Under certain approximations we show that the N-soliton interaction for the unperturbed NLS equation is described by the complex Toda chain (CTC) with N nodes, which is a completely integrable dynamical system with $2N$ degrees of freedom. A comparison made by numeric simulation shows that CTC gives an adequate description for the soliton interactions for a number of choices of the initial conditions. [S1063-651X(97)03104-8]

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I. INTRODUCTION

As is well known, the nonlinear Schrödinger equation (NLSE) serves as a basic physical model with applications in quantum mechanics, hydrodynamics, plasma physics, nonlinear optics, etc. The NLSE can be integrated by using the inverse scattering method [1]. This allows for an exhaustive study of its properties as an infinite dimensional completely integrable Hamiltonian system. Moreover, the interaction of the solitons of the NLSE in the generic case, when all solitons have different velocities is well known [1,2]. However, the cases when two or several solitons move with the same velocity [3], or when the perturbed NLSE

$$iu_t+\tfrac{1}{2}u_{xx}+|u|^2u(x,t)=iR[u] \tag{1}$$

is considered, still contain open problems. (Here we used the normalized dimensionless variables x and t; the same also holds true for the soliton parameters, plotted in the figures below.) At the same time such soliton trains moving in real media are of great interest for a number of different physical applications. Typical examples of such applications are optical soliton transmission lines [4–6] and nonlinear fiber lasers [5]. For other physical applications, see [7].

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To solve such type of problems one may use one of four methods. The first is an analytical approach based on the inverse scattering method [1,2] and expansions over "squared" solutions of the Zakharov-Shabat system [8–11] L. It allows one to relate the variations of the soliton parameters λ_k^+ and C_k^+ to integrals of the form

$$\Delta\lambda_k^+=C_k^+\int_{-\infty}^{\infty}dx\ \mathrm{tr}\{\Delta Q[u],\Psi^+(x,\lambda_k^+)\}, \tag{2a}$$

$$\Psi^+(x,\lambda)=\begin{pmatrix} 0 & -[f_1^+(x,\lambda)]^2 \\ [f_2^+(x,\lambda)]^2 & 0 \end{pmatrix}, \tag{2b}$$

$$\Delta Q[u]=\begin{pmatrix} 0 & R[u] \\ R^*[u] & 0 \end{pmatrix}, \tag{2c}$$

where $R[u]$ is the perturbation on the right-hand side of Eq. (1) and $f_1(x,\lambda_k^+)$ and $f_2(x,\lambda_k^+)$ are the components of the Jost solution $f(x,\lambda_k^+)=\binom{f_1}{f_2}$ of L, corresponding to the eigenvalue λ_k^+. Somewhat more complicated are the expressions for ΔC_k^+: the variations of the "normalization" constants of $f(x,\lambda_k^+)$. They have the same structure as Eq. (2) but also include the derivatives

$$\dot{\Psi}_k^+=\frac{d}{d\lambda}\Psi^+(x,\lambda)|_{\lambda=\lambda_k^+}\ .$$

Analogous formulas also exist for variations of the scattering data on the continuous spectrum of L. In what follows we will neglect, following [12], the influence of the continuous spectrum. This approach, known as the adiabatic approximation, is applicable for generic perturbation.

However, the corresponding formulas beyond $N=1$ quickly become so involved that one is not able to analyze them. Another problem arises from the necessity of determining the exact scattering data, corresponding to an initial pulse train of the form

$$u(x,0)=\sum_{k=1}^{N} \frac{2\nu_k e^{i\phi_{0k}}}{\cosh 2\nu_k(x-x_{0k})}. \tag{3}$$

This method has also been used for numerical simulations in several papers [13–17], where mostly two-soliton interactions have been studied. With the growth of the number of pulses the difficulties of such investigations grow enormously (see [14], where some results for $N=3$ have been obtained). Applying numerical methods in such an approach requires a multiple execution of the following procedures. First, starting from the initial condition (3), one has to determine the corresponding scattering data ($C_k^{\pm}$, $\lambda_k^{\pm}$ and the possible presence of radiation) and squared solutions $\Phi_k^{\pm}(x,\lambda)$ which enter into Eq. (2). Next, calculating the right-hand sides of Eq. (2), one obtains the evolution of the scattering data, and then one needs to determine the shape of the pulse, corresponding to the data obtained. This could possibly be simplified and used effectively in cases when the distances between the pulses are large and the number of solitons is comparatively low; see [15,16].

The second approach was initiated by the pioneering paper by Karpman and Solov'ev [12]; it is also known as the quasiparticle approach. It is based on the adiabatic approximation mentioned above. Its main idea is to view the interaction as a slow deformation of the soliton parameters in which only the nearest-neighbor interaction should be taken into account. With it, one is able to study the interaction of soliton trains for some restricted class of initial conditions, that is, (a) the solitons have nearly or exactly equal amplitudes and velocities and (b) the separation between them is large as compared to their width (more precisely these conditions will be stated in Sec. I below). Under these approximations the N-soliton solution of the NLSE and the corresponding squared solutions of the Zakharov-Shabat system are very well approximated by linear combinations of their one-soliton counterparts, and so one is able to derive a dynamical system of equations for the soliton parameters. This was performed effectively in [12] for $N=2$ solitons, where, moreover, the corresponding dynamical system was solved explicitly. Later this approach was used in a number of papers for analyzing the two-soliton interactions in the presence of various perturbations, both Hamiltonian and dissipative ones; for a review, see [4,6]. Although the region of soliton parameters to which the Karpman-Solov'ev (KS) method is applicable is comparatively small, it represents a substantial physical interest, since a great part of the experimentally studied solitonlike pulses in nonlinear fiber optics satisfy these conditions.

The third approach that can be used for analyzing the soliton interactions is the so-called variational approach proposed by Anderson and co-workers [18,19]. It is based on the Lagrangian formulation of the corresponding perturbed NLSE. To use it, one needs an ansatz for the pulse solution, thus fixing up the parametrization of the pulse. Then, inserting it into the Lagrangian one is able to derive a set of dynamical equations for the evolution of the parameters. This method is more flexible than the KS approach in the sense that a larger class of initial pulses (e.g., chirped solitons [20]) can be considered. On the other hand, it is limited by the requirement that the perturbed NLSE has to be Hamiltonian, which is not necessary for the KS method. A method proposed by Malomed [21] combines ideas from the second and third approaches. There the author derived and investigated the properties of the effective interaction Hamiltonian describing the two-soliton interactions of the perturbed NLSE and its generalization, the Ginzburg-Landau equation.

Note that practically all results obtained by the above-mentioned three analytical approaches concern the two-soliton interaction. Meanwhile it was shown that a soliton train consisting of N interacting solitons may be considered an interesting type of dynamical system with its own peculiarities [14,22–25]. Obviously the N-soliton interaction is representative of situations encountered in communication lines. The main tool for analyzing this problem was the fourth approach, based on direct numerical solving of the NLSE by the beam propagation method; for a review, see [5]. Moreover, the numerical solution of the NLSE is the main test in analyzing the applicability of the analytical techniques mentioned above.

Our aim in the present paper consists of generalizing the KS method to the case of N well separated interacting solitons with nearly or exactly equal amplitudes and velocities. In Sec. II we derive the generalization of the KS system for the N-soliton solution of the NLSE without perturbation. We prove, as was conjectured in [26,27], that the interaction is of the nearest-neighbor type.

In Sec. III we explicitly obtain the effect of three different classes of perturbations to the generalized KS system. We prove that perturbations linear in u lead only to self-interaction terms for each of the solitons separately. The perturbations cubic in u give rise not only to self-interaction terms, but also influence the interaction terms between the neighboring solitons. In Sec. III C, we analyze the driving force case and two perturbations linear in u, whose coefficients depend explicitly on x; they also lead only to self-interactive terms.

In Sec. IV we find that, under certain approximations, the N-soliton interaction for the unperturbed NLS equation is described by the complex Toda chain (CTC) with N nodes. The generic N-soliton solution, as well as the generalized Karpman-Solov'ev system (GKS) and the CTC with N nodes are dynamical systems with $2N$ degrees of freedom. The CTC is obtained from the well known real Toda chain (RTC) [28,29] by a complex extension of its dynamical variables. Numerical studies show that the RTC describes very well the positions and velocities of N interacting equidistant and out-of-phase solitons with (nearly) equal amplitudes [31]. The RTC cannot provide a description of the amplitudes and phase differences, which are assumed to be constants. CTC and GKS take into account all $4N$ soliton parameters, and one may expect that they will have wider applicability, e.g., they could be used not only for out-of-phase solitons.

It is well known that the standard RTC with real-valued dynamical variables is a completely integrable Hamiltonian system [28,29]. Some of these results for the RTC can be generalized for the CTC in a quite straightforward way. This is so for the Lax representation and the classical r-matrix method (see [2]). Using them, we show that the CTC possesses N complex-valued (or $2N$ real-valued) integrals of motion in involution, and hence is also completely integrable.

Formally the solutions of the CTC can be obtained from the ones of the RTC by taking the parameters to be complex. This means that the class of solutions of the CTC is larger, and so is the variety of their asymptotic behaviors. In addition, while all solutions of the RTC are regular for all values of t, some of the solutions of the CTC (for particular choices of the initial conditions) develop singularities for finite values of t [32]. This is so even for the simplest case $N=2$; the corresponding CTC is equivalent to the KS system which has both periodic and singular solutions.

Our result in this respect is that the CTC provides a good description for the soliton interactions for a number of different choices of the initial conditions, see Sec. IV A. This holds true even for some of the singular solutions, which are adequate for values of t outside of small region around the singularities. Another important conclusion drawn here is that the N-soliton interaction may contain principally different effects as compared to the elementary two-soliton interactions and to the RTC dynamics; see Sec. IV F.

We conclude with some conclusions and open problems. Part of the results in this paper have been previewed in [30,31,33].

II. N-SOLITON KARPMAN-SOLOV'EV SYSTEM

A. Derivation of the generalized Karpman-Solov'ev system

This section will be devoted to the NLS equation, i.e., to Eq. (1), with

$$R[u]=0. \tag{4}$$

The particular cases with linear and cubic in u perturbations, including a number of physically important ones, will be considered in the following sections.

We start by reminding the reader of the main results of Karpman and Solov'ev [12] and generalizing them to the case of N interacting solitons.

It is well known that the NLS equation is a completely integrable Hamiltonian system. It can be solved with the help of the inverse scattering method (ISM) applied to the so-called Zakharov-Shabat system:

$$Lf(x,t,\lambda)\equiv\left(i\sigma_3\frac{d}{dx}+Q(x,t)\right)f(x,t,\lambda)=\lambda f(x,t,\lambda), \tag{5}$$

where the potential $Q(x,t)$ is expressed in terms of $u(x,t)$ by

$$Q(x,t)=\begin{pmatrix} 0 & u(x,t) \\ -u^*(x,t) & 0 \end{pmatrix}. \tag{6}$$

There are many ways to derive the reflectionless potentials of L and its corresponding eigenfunctions; this immediately produces the soliton solutions of the NLS equation. In what follows below we shall need a convenient parametrization for the one soliton solution and the corresponding eigenfunction of L:

$$u_{1s}(z,t)=\frac{2\nu e^{i\phi}}{\cosh z}, \tag{7a}$$

$$z=2\nu[x-\xi(t)], \tag{7b}$$

$$\phi(z,t)=\frac{\mu}{\nu}z+\delta(t), \tag{7c}$$

$$\xi(t)=2\mu t+\xi_0, \tag{7d}$$

$$\delta(t)=2(\mu^2+\nu^2)t+\delta_0, \tag{7e}$$

$$f(x,t,\lambda)=\frac{e^{i\lambda x}}{2\cosh z}\begin{pmatrix} -ie^{-i\phi} \\ e^{z} \end{pmatrix}. \tag{7f}$$

We have denoted by $\delta(t)$ and $\xi(t)$ the soliton phase and position, respectively; δ_0 and ξ_0 determine their initial values for $t=0$; ν is the soliton amplitude and μ is its velocity.

Physically the most interesting initial configurations are those representing sum of well separated pulses with nearly equal amplitudes and velocities, that is,

$$u_0(x)=u_{Ns}(x,t=0), \tag{8}$$

$$u_{Ns}(x,t)\simeq\sum_{k=1}^{N}u_k(z_k,t), \tag{9}$$

where $u_k(z_k,t)$ is given by Eq. (7a) with z, ϕ, ξ, and δ replaced by z_k, ϕ_k, ξ_k, and δ_k, respectively.

An important paritcular case of Eq. (8) is the case in which the eigenvalues of L have equal real parts. Such an inital condition is usually referred to as an N-soliton bound state.

We stress here that generically the Zakharov-Shabat system with a potential fixed by the initial condition (8) possesses N pairs of eigenvalues $\lambda_k^{\pm}=\tilde{\mu}_k\pm i\tilde{\nu}_k$ and some nontrivial scattering data on the continuous spectrum. This fact, together with the problem of reconstructing of the spectral data of L, corresponding to initial conditions (8) and (9) for $N=2$ and 3 was analyzed in [14]. Generically, even for initial conditions with $\mu_k=0$ and $\nu_k=\nu$, we obtain eigenvalues with $\tilde{\nu}_k\neq\tilde{\nu}_j$ and $\tilde{\mu}_k\neq\tilde{\mu}_j$ for $k\neq j$, i.e., the operator L has N simple discrete eigenvalues. Most of the results obtained with the ISM have been derived precisely for this generic situation. However, the nontrivial interrelation between ν_k,μ_k and $\tilde{\nu}_k,\tilde{\mu}_k$ presents as one of the serious difficulties of the analytical approach.

To our assumptions above we add one more: that the solitons are well separated, so that their overlap is small. Then the N-soliton solution can well be approximated by the sum of N one-soliton terms as in Eq. (9) above. Mathematically these restrictions can be expressed as

$$|\mu_k-\mu_n|\ll\mu, \tag{10a}$$

$$|\nu_k-\nu_n|\ll\nu, \tag{10b}$$

$$\mu=\frac{1}{N}\sum_{k=1}^{N}\mu_k, \tag{10c}$$

$$\nu=\frac{1}{N}\sum_{k=1}^{N}\nu_k, \tag{10d}$$

$$\nu|\xi_{0k}-\xi_{0n}|\gg 1, \tag{10e}$$

$$|\nu_k-\nu_n||\xi_{0k}-\xi_{0n}|\ll 1. \tag{10f}$$

Next we insert Eq. (9) into Eq. (1) with $R[u]=0$. It is not difficult to see that, due to nonlinearity, the kth soliton will be influenced by the others. This influence will contain terms of first and second order with respect to the overlap. We will take into account only the first-order terms and suppose that their influence only changes the soliton parameters. Thus we obtain

$$iu_{k,t}+\tfrac{1}{2}u_{k,xx}+|u_k|^2u_k=iR_k^{(0)}[u]+i\tilde{R}_k^{(0)}[u], \tag{11}$$

where

$$R_k^{(0)}[u]=\sum_{n\neq k}(2|u_k|^2u_n+u_k^2u_n^*)=\sum_{n\neq k}R_{kn}^{(0)}[u], \tag{12a}$$

$$\tilde{R}_k^{(0)}[u]=\sum_{n\neq m\neq k}(2u_ku_m^*u_n+u_nu_mu_k^*). \tag{12b}$$

We prove in Appendix B that the terms in $\tilde{R}_k^{(0)}[u]$ can in fact be neglected.

Note that now we have no real perturbation; the terms $R_k^{(0)}[u]$ and $\tilde{R}_k^{(0)}[u]$ on the right-hand side of Eq. (11) just take into account the fact that we deal with an approximation to the N-soliton solution. Analogously, any additional perturbative terms on the right-hand side of Eq. (1),

$$R[u]=\sum_{p>0}R^{(p)}[u], \tag{13}$$

will lead to nontrivial contributions to the right-hand side of Eq. (11). We will denote there by $R_k^{(p)}[u]$; their explicit form will be given in Sec. III below.

Our first aim here will be to evaluate the effect of each of the summands $R_{k,n}^{(0)}$ on the right hand side of Eq. (11) on the parameters of the kth soliton. This can be done in much the same way as for the two-soliton case. Next we will also take into account the possible perturbations. In general the result can be cast into the forms

$$\frac{d\nu_k}{dt}=N_k^{(0)}[u]+\sum_{p>0}N_k^{(p)}[u], \tag{14a}$$

$$\frac{d\mu_k}{dt}=M_k^{(0)}[u]+\sum_{p>0}M_k^{(p)}[u], \tag{14b}$$

$$\frac{d\xi_k}{dt}=2\mu_k+\Xi_k^{(0)}[u]+\sum_{p>0}\Xi_k^{(p)}[u], \tag{14c}$$

$$\frac{d\delta_k}{dt}=2(\mu_k^2+\nu_k^2)+X_k^{(0)}[u]+\sum_{p>0}X_k^{(p)}[u], \tag{14d}$$

$$X_k^{(p)}[u]=2\mu_k\Xi_k^{(p)}[u]+D_k^{(p)}[u], \tag{14e}$$

where the right-hand sides of Eqs. (14a)–(14d) are determined by $R_k^{(p)}[u]$ through

$$N_k^{(p)}[u]=\tfrac{1}{2}\int_{-\infty}^{\infty}\frac{dz_k}{\cosh z_k}\,\mathrm{Re}(R_k^{(p)}[u]e^{-i\phi_k}), \tag{15}$$

$$M_k^{(p)}[u]=\tfrac{1}{2}\int_{-\infty}^{\infty}\frac{dz_k\sinh z_k}{\cosh^2 z_k}\,\mathrm{Im}(R_k^{(p)}[u]e^{-i\phi_k}), \tag{16}$$

$$\Xi_k^{(p)}[u]=\frac{1}{4\nu_k^2}\int_{-\infty}^{\infty}\frac{dz_k z_k}{\cosh z_k}\,\mathrm{Re}(R_k^{(p)}[u]e^{-i\phi_k}), \tag{17}$$

$$D_k^{(p)}[u]=\frac{1}{2\nu_k}\int_{-\infty}^{\infty}\frac{dz_k(1-z_k\tanh z_k)}{\cosh z_k}\,\mathrm{Im}(R_k^{(p)}[u]e^{-i\phi_k}), \tag{18}$$

where $p=0,1,2,\ldots$.

First we deal with $R_k^{(0)}[u]$. Inserting into Eqs. (15) and (16) the expressions for $R_{kn}^{(0)}[u]$, which due to Eqs. (7a)–(7e) and (12) take the form

$$R_{kn}^{(0)}[u]e^{-i\phi_k}=\frac{8\nu_k^2\nu_n}{\cosh^2 z_k\cosh z_n}(2e^{i(\phi_n-\phi_k)}+e^{i(\phi_k-\phi_n)}), \tag{19}$$

we find

$$N_k^{(0)}[u]=\sum_{n\neq k}\mathrm{Re}\,4\nu_k^2\nu_n[2e^{-i\phi_{0;kn}}\mathcal{P}_3(-a_{kn},b_{kn},\beta_{kn}) +e^{i\phi_{0;kn}}\mathcal{P}_3(a_{kn},b_{kn},\beta_{kn})], \tag{20}$$

$$M_k^{(0)}[u]=\sum_{n\neq k}\mathrm{Im}\,4\nu_k^2\nu_n[2e^{-i\phi_{0;kn}}\mathcal{Q}_4(-a_{kn},b_{kn},\beta_{kn}) +e^{i\phi_{0;kn}}\mathcal{Q}_4(a_{kn},b_{kn},\beta_{kn})], \tag{21}$$

where

$$\mathcal{P}_p(a,b,\beta)=\int_{-\infty}^{\infty}\frac{dz\,e^{iaz}}{\cosh^p z\,\cosh[(1+b)z+\beta]}, \tag{22}$$

$$\mathcal{Q}_p(a,b,\beta)=\int_{-\infty}^{\infty}\frac{dz\,e^{iaz}\sinh z}{\cosh^p z\,\cosh[(1+b)z+\beta]}, \tag{23}$$

and the parameters a_{kn}, b_{kn}, and β_{kn}, depending on the parameters of the kth and nth solitons, and are introduced as

$$az+\phi_0 \to \phi_k-\phi_n=a_{kn}z_k+\phi_{0;kn}, \tag{24}$$

$$bz+\beta \to z_n-z_k=b_{kn}z_k+\beta_{kn}, \tag{25}$$

$$a_{kn}=\frac{\mu_k-\mu_n}{\nu_k}, \tag{26a}$$

$$b_{kn}=\frac{\nu_n-\nu_k}{\nu_k}, \tag{26b}$$

$$\beta_{kn}=2\nu_n(\xi_k-\xi_n), \tag{26c}$$

$$\phi_{0;kn}=\delta_k-\delta_n-2\mu_n(\xi_k-\xi_n). \tag{26d}$$

The calculation of $\Xi_k^{(0)}[u]$ and $D_k^{(0)}[u]$, which contain an additional factor of z in the integrand, requires a knowledge of the first derivatives of $\mathcal{P}_p$ and $\mathcal{Q}_p$ with respect to a.

The calculation of these integrals is described in Appendix A. Most of these integrals cannot be expressed in terms of elementary functions. Even if that was possible, we would have obtained an overcomplicated system of equations, which could hardly be used for practical calculations. Therefore we will limit ourselves to the limit of these integrals for $a_{kn}\to 0$, $b_{kn}\to 0$, and $\beta_{kn}\gg 1$. Our reasons for this are the following: first, as is shown in Appendix A, the precise answers for these integrals are smooth and well behaved functions for small values of b_{kn} and a_{kn}. The second reason is that we have already neglected terms of the same order as we could account for.

As a result, for $\beta\gg 1$ we find

$$\mathcal{P}_p(a,b,\beta)\simeq 4e^{-|\beta|}\left(1-\frac{ias_\beta}{p-1}\right)\mathcal{J}_{p-1}(a), \tag{27a}$$

$$\mathcal{Q}_p(a,b,\beta)\simeq \frac{4e^{-|\beta|}\mathcal{A}_p(a,\beta)}{(p-1)(p-2)}\mathcal{J}_{p-2}(a), \tag{27b}$$

$$\mathcal{A}_p(a,\beta)=s_\beta a^2+i(p-1)a-s_\beta(p-2), \tag{27c}$$

$$s_\beta=\operatorname{sgn}\beta, \tag{27d}$$

where by $\mathcal{J}_p(a)$ we have denoted the integrals

$$\mathcal{J}_p(a)=\int_{-\infty}^{\infty}\frac{dz\, e^{iaz}}{2\cosh^p z}=\int_0^\infty\frac{dx\cos ax}{\cosh^p x}. \tag{28}$$

Obviously, $\mathcal{J}_p(a)$ are even functions of a, while their derivatives $\mathcal{J}_p'(a)=(d\mathcal{J}_p/da)(a)$ are odd functions of a. In particular, their values for $a=0$ are given by

$$\mathcal{J}_1(0)=\frac{\pi}{2}, \tag{29a}$$

$$\mathcal{J}_2(0)=1, \tag{29b}$$

$$\left.\frac{d\mathcal{J}_p(a)}{da}\right|_{a=0}=0, \tag{29c}$$

$$\mathcal{J}_{2n}(0)=\frac{(2n-2)!!}{(2n-1)!!}, \tag{29d}$$

$$\mathcal{J}_{2n+1}(0)=\frac{(2n-1)!!}{(2n)!!}\,\frac{\pi}{2}. \tag{29e}$$

Here we keep the a dependence because, in calculating $\Xi_k^{(0)}[u]$ and $D_k^{(0)}[u]$, we will need the derivatives of $\mathcal{P}_p(a,b,\beta)$ and $\mathcal{Q}_p(a,b,\beta)$ with respect to a.

Now it is not difficult, using Eqs. (27)–(29), to calculate the contribution of each summand (19) to the right-hand sides of Eqs. (14a)–(14d) and to derive the following set of $4N$ equations, generalizing the Karpman-Solov'ev system:

$$\frac{d\nu_k}{dt}=\sum_{n\neq k}16\nu_k^2\nu_n e^{-|\beta_{kn}|}\sin\phi_{0;kn}, \tag{30}$$

$$\frac{d\mu_k}{dt}=-\sum_{n\neq k}16\nu_k^2\nu_n s_{kn}e^{-|\beta_{kn}|}\cos\phi_{0;kn}, \tag{31}$$

$$\frac{d\xi_k}{dt}=2\mu_k-\sum_{n\neq k}4\nu_n s_{kn}e^{-|\beta_{kn}|}\sin\phi_{0;kn}, \tag{32}$$

$$\frac{d\delta_k}{dt}=2(\mu_k^2+\nu_k^2)+\sum_{n\neq k}(-8\mu_k\nu_n s_{kn}e^{-|\beta_{kn}|}\sin\phi_{0;kn}+24\nu_k\nu_n e^{-|\beta_{kn}|}\cos\phi_{0;kn}). \tag{33}$$

Here the summation is over the nearest neighbors of the kth soliton, i.e., the ones for which $\xi_k-\xi_n$ is minimal. Indeed, since we care only about terms of order $e^{-|\beta_{kn}|}$, taking into account the other terms will be an overestimated precision.

It is worth noting, that the number of the summands in Eqs. (30)–(33) will depend very much on the initial configuration of the system. In what follows, we suppose that the solitons form a chainlike configuration of nearly equidistant solitons, and that the kth soliton has as its nearest neighbors the $k-1$st and $k+1$st; then each such sum for $1<k<N$ will contain only two terms, while only one for $k=1$ and $k=N$. We can also assume without restrictions that $\xi_k<\xi_{k+1}$; then $s_{k,k-1}=1$ and $s_{k,k+1}=-1$. In this case, after introducing the notations

$$S_{k,n}=e^{-|\beta_{kn}|}\nu_n\sin s_{kn}\phi_{0;kn}, \tag{34a}$$

$$C_{k,n}=e^{-|\beta_{kn}|}\nu_n\cos\phi_{0;kn}, \tag{34b}$$

we find that the system (30)–(33) can be rewritten as

$$\frac{d\nu_k}{dt}=16\nu_k^2(S_{k,k-1}-S_{k,k+1}), \tag{35}$$

$$\frac{d\mu_k}{dt}=-16\nu_k^2(C_{k,k-1}-C_{k,k+1}), \tag{36}$$

$$\frac{d\xi_k}{dt}=2\mu_k-4(S_{k,k-1}+S_{k,k+1}), \tag{37}$$

$$\frac{d\delta_k}{dt}=2(\mu_k^2+\nu_k^2)-8\mu_k(S_{k,k-1}+S_{k,k+1})$$
$$+24\nu_k(C_{k,k-1}+C_{k,k+1}). \tag{38}$$

B. Discussion

System (35)–(38) is rather complicated as it is, and therefore we have two possibilities to state some properties of its solutions: (i) to solve it numerically and (ii) to consider some particular cases as well as some further approximations. In order to estimate the performance of system (35)–(38), we solve it numerically for different initial conditions and compare the obtained results with these from the numerical solution of the NLSE (1) by the beam propagation method (BPM) [31]. A very good agreement has been identified for the case of initially equal and equidistant pulses with initial phase difference 0 and π. Moreover, it has been shown that system (35)–(38) could be useful even in describing the interaction of pulses with initially unequal amplitudes. Indeed, in [34] a numerical comparison between the GKS and BPM has been performed for the case of four equidistant solitons with alternatively changing amplitudes. It has been shown that there is very good agreement (an error less than 3%) between them for distances from $r_0=5$ to 10. Several remarks are in order.

(1) Up to now we have not considered physical perturbations. However, the approximation itself violates the integrability of the NLS and leads to the highly nonlinear GKS.

(2) The right-hand sides of the GKS contain two types of terms. The first type describes the soliton self-interaction (see [35]). The second type of terms is characteristic of the two-soliton interactions [12]. These terms relate only the nearest neighbors of the solitons.

(3) Since the GKS is a nonlinear system, then it *does not* allow superposition principle. Therefore it is not possible, knowing the two-soliton interactions, to describe the interactions of $N\geqslant 3$ solitons; indeed a middle soliton would be influenced by its left and right neighbors and its behavior would be very different from one of the end solitons.

(4) The N-soliton system has $2N$ degrees of freedom; its behavior is determined generically by $4N$ real constants, fixing up its initial condition. Of course such systems would have a much wider class of solutions that cannot be reduced to the two-soliton case. An example showing that this is really so is provided in Sec. IV F below.

(5) In Sec. IV we derive an integrable approximation to the GKS, and show that it can be useful for a larger class of initial conditions.

III. PERTURBED *N*-SOLITON KARPMAN-SOLOV'EV SYSTEM

In this section we shall describe the effects to the generalized KS system due to the presence of various perturbation terms $R^{(p)}[u]$ on the right hand side of Eq. (1). In the first two subsections we consider generic perturbations linear and cubic in u, with complex nonvanishing coefficients. Next we deal with perturbations relevant for the phase and amplitude modulations and the driving force case. Section III D is devoted to a brief discussion of the physical meaning of the perturbations for various choices of the constants c_s and d_s in Eqs. (39) and (44).

A. Case with linear in u perturbations

We start with the case when

$$R^{(1)}[u]=\sum_{s=0}^{3} c_s\frac{d^s u}{dx^s}, \tag{39}$$

where $c_s=c_{s0}+ic_{s1}$ are complex constants. Fixing up their values in a convenient way, we can describe a number of physically important perturbations. We shall discuss these point in Sec. III D.

Inserting these terms into the right hand sides of Eqs. (15)–(18) after somewhat lengthy calculations, we obtain that they lead to the following additional terms in system (14a)–(14d):

$$N_k^{(1)}[u]=2c_{00}\nu_k-4c_{11}\mu_k\nu_k-8c_{20}\nu_k\left(\frac{\nu_k^2}{3}+\mu_k^2\right)$$
$$+16c_{31}\nu_k\mu_k(\mu_k^2+\nu_k^2), \tag{40}$$

$$M_k^{(1)}[u]=-\frac{4}{3}(c_{11}\nu_k^2+4c_{20}\mu_k\nu_k^2)+16c_{31}\nu_k^2(\mu_k^2+\tfrac{7}{15}\nu_k^2), \tag{41}$$

$$\Xi_k^{(1)}[u]=-c_{10}+4c_{21}\mu_k+4c_{30}(3\mu_k^2+\nu_k^2), \tag{42}$$

$$X_k^{(1)}[u]=c_{01}+4c_{21}(\mu_k^2-\nu_k^2)+16c_{30}\mu_k(\mu_k^2-\nu_k^2). \tag{43}$$

As is obvious from the above system, the perturbations linear in u contribute only terms which are local in k. It is natural to call such terms self-interactive.

B. Case with cubic in *u* perturbations

In this subsection we consider three types of cubic perturbations

$$R^{(2)}[u]=d_0|u|^2u+\frac{d_1}{4}u(|u|^2)_x+\frac{d_2}{4}(|u|^2u_x-u_x^*u^2), \tag{44}$$

where again

$$d_s=d_{s0}+id_{s1} \tag{45}$$

are complex constants. The calculations are similar to those in Sec. III A, although more involved. In particular, due to the presence of derivative terms, some other types of integrals appear (see the Appendixes). However, in our approximations they can be explicitly evaluated. The result here is given by

$$N_k^{(2)}[u]=\frac{16\nu_k^3}{3}(d_{00}-d_{21}\mu_k)+16\nu_k^2\sum_{n\neq k}\left\{\left[3d_{00}-d_{21}(2\mu_k+\mu_n)-\frac{\nu_k}{3}d_{10}s_{kn}\right]C_{kn}\right.$$
$$\left.+\left[d_{01}s_{kn}+d_{10}(\mu_n-\mu_k)s_{kn}+d_{20}\mu_k s_{kn}+\frac{\nu_k}{3}(d_{11}-4d_{21})\right]S_{kn}\right\},\tag{46}$$

$$M_k^{(2)}[u]=-\frac{16\nu_k^4}{15}d_{11}+\frac{16\nu_k^2}{3}\sum_{n\neq k}\{-[3d_{01}s_{kn}+d_{11}\nu_k+d_{20}(2\mu_k+\mu_n)s_{kn}]C_{kn}$$
$$+[d_{00}+(d_{11}-2d_{20})\nu_k s_{kn}+d_{11}(2\mu_k-\mu_n)-d_{21}\mu_k]S_{kn}\},\tag{47}$$

$$\Xi_k^{(2)}[u]=-\frac{2\nu_k^2}{3}d_{10}-4\sum_{n\neq k}\{[3d_{00}s_{kn}+d_{10}\nu_k-d_{21}(2\mu_k+\mu_n)s_{kn}]C_{kn}$$
$$+[d_{01}+(d_{11}-2d_{21})\nu_k s_{kn}+d_{10}(\mu_n-\mu_k)+d_{20}\mu_k]S_{kn}\},\tag{48}$$

$$X_k^{(2)}[u]=\frac{4\nu_k^2}{3}[3d_{01}+\mu_k(3d_{20}-d_{10})]+8\sum_{n\neq k}\{[3(d_{01}\nu_k-d_{00}\mu_k s_{kn})+(d_{20}\nu_k+d_{21}\mu_k s_{kn})(2\mu_k+\mu_n)$$
$$-\nu_k(d_{10}\mu_k+d_{11}\nu_k s_{kn})]C_{kn}-[(d_{01}+d_{10}(\mu_n-\mu_k)+d_{20}\mu_k)\mu_k$$
$$+(d_{00}+d_{11}(2\mu_k-\mu_n)-d_{21}\mu_k-d_{20}\nu_k s_{kn})\nu_k s_{kn}]S_{kn}\}.\tag{49}$$

C. Pseudolinear types of perturbations

Here we present results for some special type of perturbations, which also present physical interest. These include

$$R^{(3)}[u]=(\tilde{c}x+\delta)u(x,t)+\beta u_{xx},\tag{50}$$

which is important for the effects of sliding filters [16], the driving force perturbation

$$R^{(4)}=f_0e^{i\Omega x}+f_1e^{-i\Omega x},\tag{51}$$

where we consider Ω to be a real and f_0 and f_1 complex constants, and

$$R^{(5)}[u]=(f_0e^{i\Omega x}+f_1e^{-i\Omega x})u(x,t),\tag{52}$$

which describes the phase and amplitude modulation effects.

Skipping the calculations we present the results

$$N_k^{(3)}[u]=2\tilde{c}_0\nu_k\xi_k(t)-8\beta\nu_k\left(\frac{\nu_k^2}{3}+\mu_k^2\right)+2\delta\nu_k,\tag{53}$$

$$M_k^{(3)}[u]=\tfrac{1}{2}\tilde{c}_1-\tfrac{16}{3}\beta\mu_k\nu_k^2,\tag{54}$$

$$\Xi_k^{(3)}[u]=\frac{\pi^2}{8\nu_k^2}\tilde{c}_0,\tag{55}$$

$$X_k^{(3)}[u]=-\frac{\pi^2}{4\nu_k^2}\mu_k\tilde{c}_0+\tilde{c}_1\xi_k(t),\tag{56}$$

$$N_k^{(4)}[u]=\frac{\pi}{2\cosh\omega_k}\mathrm{Re}F_+(\Omega\xi_k),\tag{57}$$

$$M_k^{(4)}[u]=\frac{\omega_k}{\cosh\omega_k}\mathrm{Re}F_-(\Omega\xi_k),\tag{58}$$

$$\Xi_k^{(4)}[u]=-\frac{\pi^2}{8\nu_k^2}\frac{\sinh\omega_k}{\cosh^2\omega_k}\mathrm{Im}F_-(\Omega\xi_k),\tag{59}$$

$$X_k^{(4)}[u]=\frac{\pi^2}{8\nu_k^2}\frac{\sinh\omega_k}{\cosh^2\omega_k}\mathrm{Im}\{F_+(\Omega\xi_k)-2\mu_kF_-(\Omega\xi_k)\},\tag{60}$$

where $\omega_k=\pi\Omega/4\nu_k$ and

$$F_\pm(\Omega\xi_k)=(f_0e^{i\xi_k\Omega}\pm f_1e^{-i\xi_k\Omega}),\tag{61}$$

and

$$N_k^{(5)}[u]=\frac{2\nu_k\omega_k}{\sinh\omega_k}\mathrm{Re}F_+(\Omega\xi_k),\tag{62}$$

$$M_k^{(5)}[u]=\frac{2\nu_k}{\pi}\frac{\omega_k^2}{\sinh\omega_k}\mathrm{Re}F_-(\Omega\xi_k),\tag{63}$$

$$\Xi_k^{(5)}[u]=\frac{\pi}{2\nu_k}\frac{1-\omega_k\coth\omega_k}{\sinh\omega_k}\mathrm{Im}F_-(\Omega\xi_k),\tag{64}$$

$$X_k^{(5)}[u]=\frac{\omega_k^2\coth\omega_k}{\sinh\omega_k}\mathrm{Im}F_+(\Omega\xi_k)$$
$$+\frac{\pi\mu_k}{2\nu_k}\frac{1-\omega_k\coth\omega_k}{\sinh\omega_k}\mathrm{Im}F_-(\Omega\xi_k).\tag{65}$$

D. Discussion of the physical relevance of the perturbations

The class of perturbations listed above is a very large one. Here we (a) briefly discuss why some of the constants c_{sk} and d_{sk}, $k=0$ and 1, can be set to 0 without a loss of generality and (b) mention special choices of the coefficients c_{sk} and d_{sk} that correspond to some physically important perturbations.

Let us consider the case when

$$i\tilde{R}[u]=ic_{10}u_x-c_{21}u_{xx}-d_{01}|u|^2u. \tag{66}$$

It is well known that such types of additional terms do not violate the integrability of the NLS equation. Indeed, after the change of variables

$$\xi=\frac{x+c_{10}t}{\sqrt{1+2c_{21}}}, \tag{67a}$$

$$\tilde{u}(\xi,t)=\sqrt{1+d_{01}}u(x,t), \tag{67b}$$

the NLS for $u(x,t)$ with the right-hand side given by Eq. (66) will go into the standard NLS equation for $\tilde{u}$ in terms of the variables ξ and t. Therefore, from now on we shall suppose that $c_{10}=c_{21}=d_{01}=0$. Of course this is possible only if $1+2c_{21}>0$ and $1+d_{01}>0$. But since we assume that these are perturbation terms, then these constants should be small and the conditions should be satisfied.

In a slightly different way we can absorb the terms

$$i\tilde{\tilde{R}}[u]=-c_{01}u+ic_{30}(u_{xxx}+6|u|^2u_x). \tag{68}$$

Indeed, $\tilde{\tilde{R}}[u]$ can be viewed as the variational derivative of two of the integrals of motion of the NLS. Therefore adding these terms to the right hand side of the NLS we will obtain one of the higher NLS-type equations with the dispersion law $\tilde{f}(\lambda)=-2\lambda^2+c_{01}+4c_{30}\lambda^3$. It is well known that the KS method can also be applied to any of the higher analogs of the NLS [11]. Of course this is possible only if the relation $d_{10}=6c_{30}$ holds. If such a relation does not hold, we have the following options: (a) consider $(d_{10}-6c_{30})|u|^2u_x$ as a perturbation term to the nonlinear evolution equation (NLEE) with dispersion $\tilde{f}(\lambda)=-2\lambda^2+c_{00}+4c_{30}\lambda^3$, (b) consider $(c_{30}-d_{10}/6)u_{xxx}$ as a perturbation term to the NLEE with dispersion $\tilde{f}(\lambda)=-2\lambda^2+c_{00}+2d_{10}\lambda^3/3$, or (c) consider $c_{30}u_{xxx}+d_{10}|u|^2u_x$ as perturbation terms to the NLS with dispersion $\tilde{f}(\lambda)=-2\lambda^2+c_{00}$. Each of these approaches gives compatible systems of evolution equations for the soliton parameters.

Without going into further details (see [4,6]), we summarize some of the physically important choices for $iR[u]$ in Table I. The papers in the second column are those in which the two-soliton interaction has been analyzed in the presence of the corresponding perturbation by the Karpman-Solov'ev technique.

It also seems that perturbations with fourth-order derivatives will hardly be needed, so we have limited ourselves to Eq. (39). Some typical cubic perturbations such as the nonlinear gain have been analyzed for the two-soliton case in [36].

As in the unperturbed case, the structure of the generalized perturbed KS system is determined by the one- and two-soliton interactions. Indeed, the right-hand sides of the equations, describing the evolution of the kth soliton, are obtained by adding (with the corresponding signs) two types of terms: (i) the self-interacting terms, which are typical for the one-soliton perturbations (see, e.g., [35,12,16]), and (ii) the interaction terms, again are of the nearest-neighbor interaction (NNI) type, that is, the kth soliton interacts with $k-1$ and $k+1$ solitons, and the corresponding terms are typical of the two-soliton interactions. Of course, the first and the last solitons in the chain have only one nearest neighbor. In view of this we remark that (a) the perturbations $iR[u]$ that are linear in u contribute only to the self-interacting terms; (b) the perturbations $iR[u]$ that are nonlinear in u influence both types of terms in the GKS; and (c) as in the unperturbed case, the perturbed GKS system is a highly nonlinear one and does not allow for a superposition principle; thus the knowledge of the two-soliton interaction cannot give us insight into the N-soliton dynamics with $N\geqslant 3$.

TABLE I. Physically important choices for the perturbations.

Nonvanishing constants	Physical phenomena
c_{00}	linear loss and/or gain [15,17]
c_{00}, c_{20}	bandwidth limited amplification [15–17,42,37]
c_{30}	third-order dispersion [38,43]
d_{00}	nonlinear loss and/or gain [42,36,16,48]
d_{11}	soliton self-frequency shift [44,16]
$d_{10}=3d_{20}$	self steepening [44]
$R^{(3)}$	sliding filters [45,16]
$R^{(5)}$ with $f_0=f_1=-i\alpha/2$, α real	phase modulation [46,47,43]

As we mentioned in Sec. II, the solitons interact even when $iR[u]=0$. Indeed, the approximation itself violates the integrability and leads to nontrivial effective perturbative terms, see Eqs. (11) and (12). Although exponentially small, the presence of the NNI terms may lead to sizable effects: see how the relative spread $\beta_N(t)$,

$$\beta_N(t)=\frac{\xi_N(t)-\xi_1(t)}{\xi_N(0)-\xi_1(0)}, \tag{69}$$

describing the divergence of the soliton train, falls off with the increase of the soliton number (see Fig. 1 and Table II).

Let us consider a typical for the fiber optics perturbation: the third-order dispersion (TOD). Such perturbation is relevant for at least two physical situations: (a) when the carrier wavelength is near the zero-dispersion wavelength of the fiber and/or (b) when the pulse width is very short.

The effect of TOD on the interaction of solitons with equal amplitudes was investigated in [14,38]. It has been proposed in [14] that the effect of TOD can be used to avoid the coalescence of two equal amplitude in-phase solitons. It

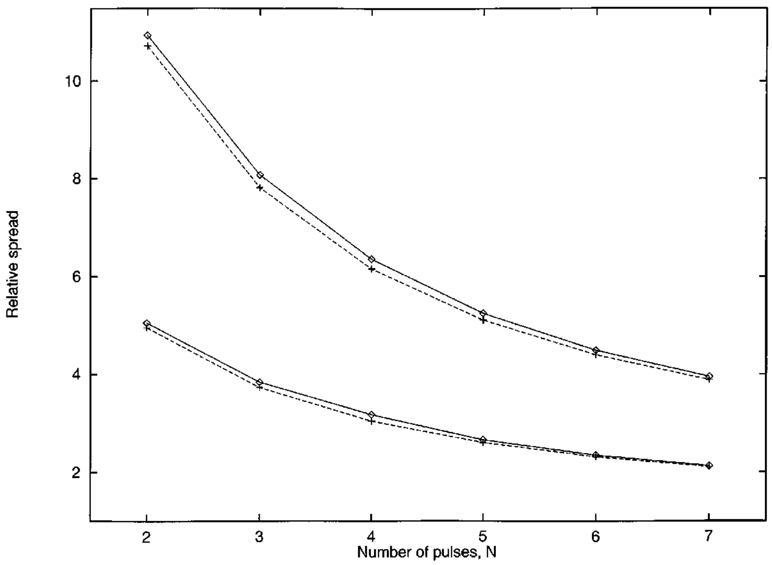

FIG. 1. Dependence of the relative spread $\beta_N(t)$ on the number of solitons N in a train with initially equal, equidistant, and out-of-phase pulses. $2\nu_k(0)=1$, $\mu_k(0)=0$, $\delta_{k+1}(0)-\delta_k(0)=\pi$, $r(0)=6$, and $t=126$ (lower curves) and $t=300$ (upper curves). Solid lines, BPM; dashed lines, CTC.

has been shown that the interaction of two solitons with equal amplitudes can be viewed as a break up of the corresponding two-soliton bound state. Further, this break up of the two-soliton bound state has been succesfully described by the two-soliton Karpman-Solov'ev soliton perturbation theory [38]. Moreover, the N-soliton interaction of solitons with equal amplitudes was also viewed as a breakup of the corresponding N-soliton bound state [38]. In the latter case acquired velocities are such that the solitons cannot only separate but also coalesce after some propagation distance. Therefore TOD does not suffice to stabilize a multisoliton train [37,38]. From the point of view of applications this is an important conclusion.

Similar effects on the N-soliton interaction can be expected also from intrapulse Raman scattering. Thus we have derived the generalized KS system in the presence of rather general perturbation terms. The system itself for $N>2$ is rather complicated, and cannot be solved analytically. Here we can repeat all our remarks from Sec. II B.

We see two important uses for the GKS. It can be used in numerical investigations of soliton interactions. An important question here is to describe the domain of initial conditions, for which the perturbed GKS is applicable. Second, after some additional approximations, this system can also be treated analytically. We show this in Sec. IV.

TABLE II. Dependence of $\beta_N(t)$ for N out-of-phase solitons for three different values of t, $N=1,\ldots,7$ and $r_0=6$ and 8. The theoretical values are evaluated from formula (120), the numerical are from BPM.

t	$t=48$		$t=126$		$t=300$	
N	Num.	Theor.	num.	theor.	num.	theor.
			$r_0=6$			
2	2.404	2.362	5.049	4.951	10.947	10.726
3	1.937	1.896	3.837	3.726	8.074	7.810
4	1.664	1.642	3.116	3.036	6.350	6.151
5	1.495	1.490	2.657	2.604	5.242	5.105
6	1.393	1.393	2.346	2.313	4.488	4.394
7	1.327	1.328	2.123	2.105	3.948	3.883
			$r_0=8$			
2	1.278	1.274	1.990	1.981	3.593	3.574
3	1.159	1.158	1.652	1.643	2.790	2.769
4	1.107	1.106	1.465	1.460	2.332	2.317
5	1.080	1.080	1.352	1.351	2.044	2.034
6	1.064	1.064	1.282	1.281	1.851	1.844
7	1.052	1.052	1.234	1.234	1.713	1.709

IV. SOLITON INTERACTIONS AND THE COMPLEX TODA CHAIN

Here we shall introduce additional simplification of system (35)–(38), which allows us to derive some analytical results about the asymptotic behavior of its solutions. The first approximation to Eqs. (35)–(38), considered in [31], consisted of the use of the average amplitude ν and the average velocity μ (instead of ν_n and μ_n) in the exponentially small terms $S_{k,n}$ and $C_{k,n}$. This follows the original idea of the Karpman-Solov'ev approach [12]. Even after that, however, system (35)–(38) remains unsolvable.

At the same time it was conjectured in [26,27] that the standard real Toda chain (RTC) (with $N=\infty$) may reasonably well describe the dynamics of the positions (in an infinite train) of soliton pulses. Such a conjecture, however, requires that the phase difference between the neighboring solitons be constant. Looking at Eq. (38) and also at the results of the numeric simulations we find that this is not the case.

Although its derivation is mathematically not consistent (see [31]), it has been established that at least for some types of initial conditions (equal and initially out of phase soliton pulses) the RTC gives a fairly good description of the pulse positions in comparison with direct solving of NLSE (1) by the beam propagation method. This we explain by the fact that for small values of t the phase difference stays rather close to constant. What happens next is that the soliton positions rather quickly tend to their asymptotic regime. In it the distances between the nearest neighbors increase exponentially, and as result the terms $S_{k,n}$ and $C_{k,n}$ become so

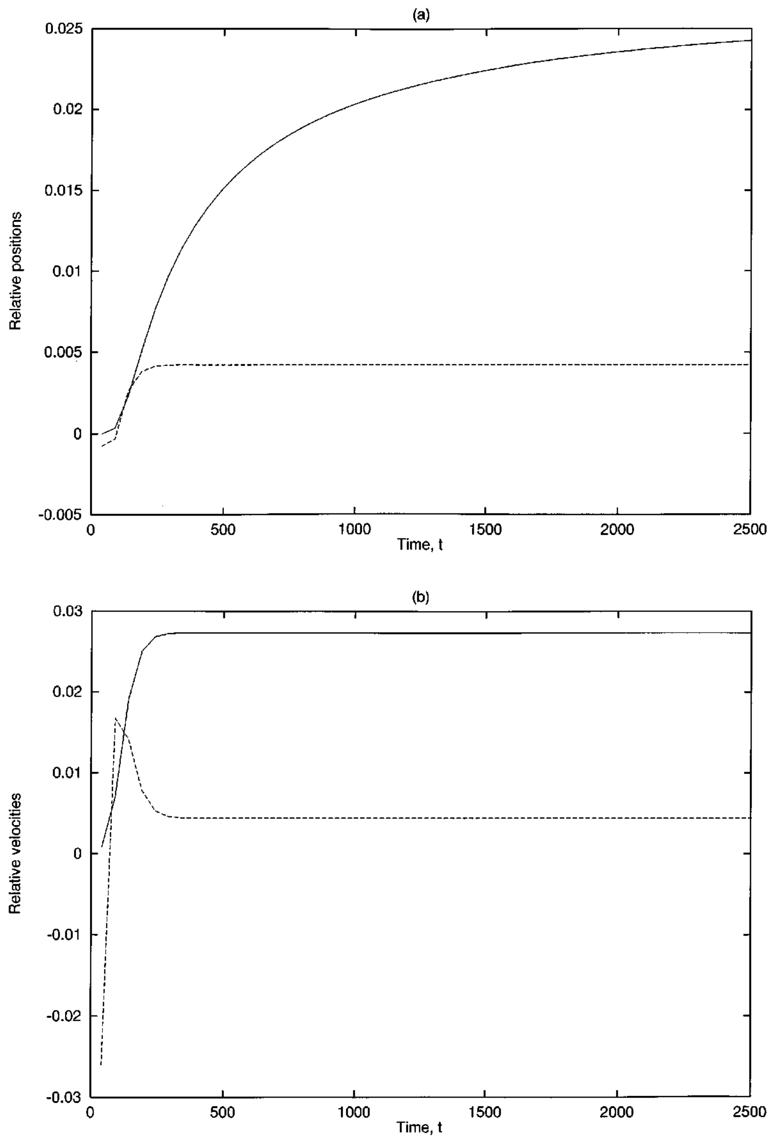

FIG. 2. Relative distances, velocities, amplitudes, and phase differences evaluated for a train of four pulses with initially equal, equidistant, and out-of-phase pulses $[2\nu_k(0)=1,\ \mu_k(0)=0$, and $\delta_{k+1}(0)-\delta_k(0)=\pi]$. We plot the ratios $[\xi_k(t)-\widetilde{\xi}_k(t)]/\xi_k(t)$, $[\nu_k(t)-\widetilde{\nu}_k(t)]/\nu_k(t)$, $[\mu_k(t)-\widetilde{\mu}_k(t)]/\mu_k(t)$, and $[\delta_k(t)-\widetilde{\delta}_k(t)]/\delta_k(t)$ where $k=1,2$ and $\widetilde{\xi}_k$, $\widetilde{\mu}_k$, $\widetilde{\nu}_k$ and $\widetilde{\delta}_k$ are the solutions for the CTC with four nodes, and ξ_k, μ_k, ν_k, and δ_k are the solutions for the GKS with $N=4$. The solid lines correspond to $k=1$, and the dashed ones to $k=2$. (a) Relative distances $[\xi_k(t)-\widetilde{\xi}_k(t)]/\xi_k(t)$. (b) Relative velocities $[\mu_k(t)-\widetilde{\mu}_k(t)]/\mu_k(t)$. (c) Relative amplitudes $[\nu_k(t)-\widetilde{\nu}_k(t)]/\nu_k(t)$. (d) Relative phase differences $[\delta_k(t)-\widetilde{\delta}_k(t)]/\delta_k(t)$.

small, that the phase difference does not have any influence.

The aim of this section is to propose another approximation to system (35)–(38), which will reduce it to the CTC with N nodes. This we view as a natural generalization of the conjecture in [26,27].

A. Derivation of the CTC

Let us make the following approximations to system (35)–(38). First we change ν_n to ν and μ_n to μ in all terms that contain S_{kn} and C_{kn}. Indeed, both S_{kn} and C_{kn} can be estimated by $|S_{kn}|\leqslant\epsilon$, $|C_{kn}|\leqslant\epsilon$, where $\epsilon=e^{-2\nu r_0}$ and $r_0=(\xi_k-\xi_{k+1})|_{t=0}$. The parameter ϵ determines the overlap between the neighboring solitons, and up to now we have been taking into account only terms of first order with respect to ϵ. Then obviously, with the above approximation we neglect terms like $|\nu-\nu_k|\epsilon$ and $|\mu-\mu_k|\epsilon$, which due to condition (10) will be of higher order.

Second, in Eqs. (37) and (38) we neglect the terms S_{kn} and C_{kn} as compare to μ_k, ν_k^2, and μ_k^2. The numerical study of the system for $N=2$, 3, and 4 with initial conditions $r_0=8$, $\phi_0=\pi$, $\mu_k(0)=0$, and $\nu_k(0)=\frac{1}{2}$ shows that $|(\widetilde{\xi}_k-\xi_k)/\xi_k|\leqslant 0.03$, i.e., smaller than 3%, where $\widetilde{\xi}_k$ is the solution of the approximated equation, while ξ_k is the solution of the exact one. Analogical study of the phases δ_k, velocities μ_k, and amplitudes ν_k show errors on the order less than 3%; see Fig. 2.

As a result the system of equations (35)–(38) goes into

$$\frac{d\nu_k}{dt}=16\nu^2(\widetilde{S}_{k,k-1}-\widetilde{S}_{k+1,k}), \tag{70}$$

$$\frac{d\mu_k}{dt}=-16\nu^2(\widetilde{C}_{k,k-1}-\widetilde{C}_{k+1,k}), \tag{71}$$

$$\frac{d\xi_k}{dt}=2\mu_k, \tag{72}$$

$$\frac{d\delta_k}{dt}=2(\mu_k^2+\nu_k^2), \tag{73}$$

where

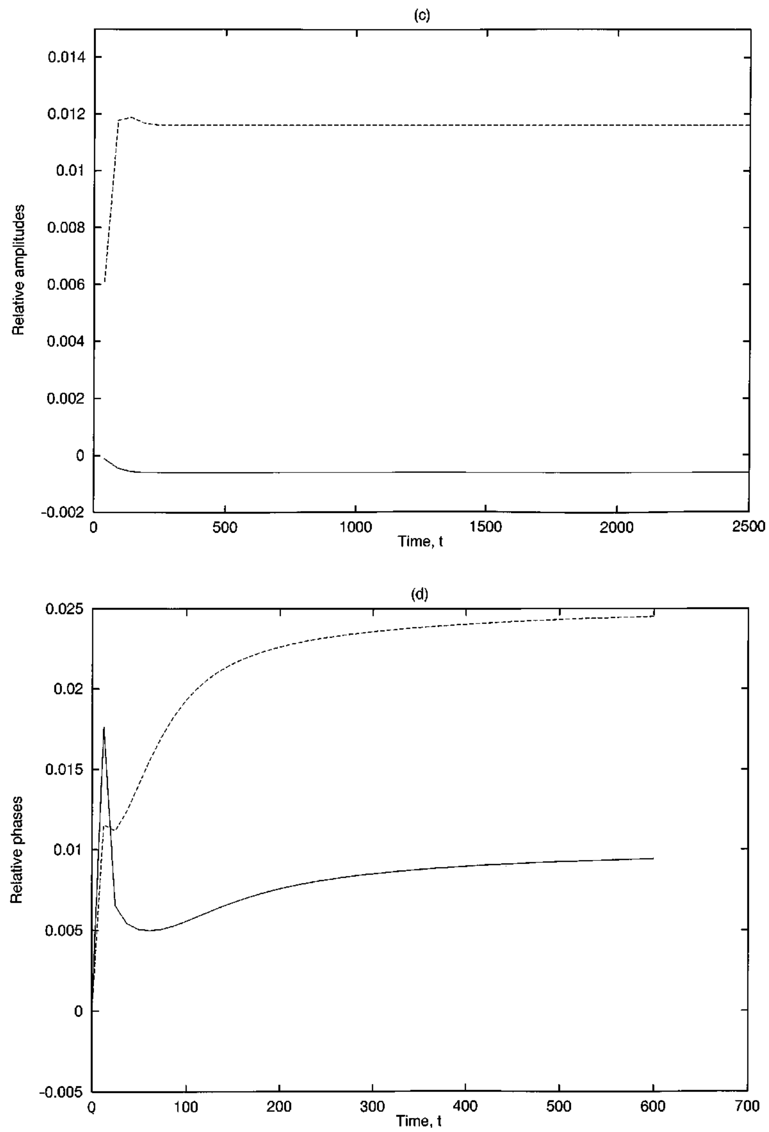

FIG. 2 (*Continued*).

$$\widetilde{C}_{k,n}=e^{-|2\nu(\xi_k-\xi_n)|}\nu\ \cos\widetilde{\phi}_{k,n}\,, \tag{74}$$

$$\widetilde{S}_{k,n}=e^{-|2\nu(\xi_k-\xi_n)|}\nu\ \sin s_{kn}\widetilde{\phi}_{k,n}\,, \tag{75}$$

$$\widetilde{\phi}_{k,n}=\delta_k-\delta_n-2\mu(\xi_k-\xi_n), \tag{76}$$

and $s_{k,k-1}=-s_{k,k+1}=1$.

Let us now introduce the function

$$\begin{aligned}E_{k,n}&=4\nu(\widetilde{C}_{k,n}-i\widetilde{S}_{k,n})\\&=\exp(-2\nu|\xi_k-\xi_n|-is_{k,n}\widetilde{\phi}_{k,n}+\ln 4\nu^2),\end{aligned} \tag{77}$$

and the complex variables

$$\lambda_k=\mu_k+i\nu_k\,. \tag{78}$$

Then the first two equations in system (70) and (71) can be rewritten as

$$\frac{d\lambda_k}{dt}=4\nu(E_{k+1,k}-E_{k,k-1}). \tag{79}$$

Let us now evaluate the derivative $dE_{k,k-1}/dt$, using on the way the above-mentioned approximations. After some calculations we obtain

$$\frac{dE_{k,k-1}}{dt}=-4\nu(\lambda_k-\lambda_{k-1})E_{k,k-1}\,. \tag{80}$$

Analogously for $E_{k+1,k}$ we have

$$\frac{dE_{k+1,k}}{dt}=-4\nu(\lambda_{k+1}-\lambda_k)E_{k+1,k}\,. \tag{81}$$

From these equations we conclude that $E_{k,k-1}$ and $E_{k+1,k}$ can be written in the form

$$E_{k,k-1}=-\exp(q_k-q_{k-1}), \tag{82a}$$

$$E_{k+1,k}=-\exp(q_{k+1}-q_k), \tag{82b}$$

where the minus sign in front of the exponents is for later convenience. The dynamical variables q_k satisfy the equations

$$\frac{dq_{k+1}}{dt}-\frac{dq_k}{dt}=-4\nu(\lambda_{k+1}-\lambda_k), \tag{83}$$

or, equivalently,

$$\frac{dq_k}{dt}=-4\nu\lambda_k. \tag{84}$$

If we now change the time variable from t to $\tau=4\nu t$ and assume $p_k=-\lambda_k$ then system (70)–(73) acquires the form

$$\frac{dp_k}{d\tau}=e^{q_{k+1}-q_k}-e^{q_k-q_{k-1}}, \tag{85}$$

$$\frac{dq_k}{d\tau}=p_k, \tag{86}$$

which is equivalent to the complex Toda chain

$$\frac{d^2q_k}{d\tau^2}=e^{q_{k+1}-q_k}-e^{q_k-q_{k-1}}. \tag{87}$$

Let us briefly discuss how the new variables p_k and q_k are related to the old ones μ_k, ν_k, ξ_k, and δ_k. Indeed, from the old variables we immediately construct the new ones. Slightly more involved are the expressions for q_k and $q_{k+1}-q_k$:

$$q_k=-2\nu\xi_k+k\ \ln 4\nu^2-i(\delta_k+\delta+k\pi-2\mu\xi_k), \tag{88}$$

$$q_{k+1}-q_k=-2\nu(\xi_{k+1}-\xi_k)+\ln 4\nu^2 -i[\delta_{k+1}-\delta_k+\pi-2\mu(\xi_{k+1}-\xi_k)], \tag{89}$$

where we have put

$$\delta=\frac{1}{N}\sum_{k=1}^{N}\delta_k. \tag{90}$$

Then both expressions in Eq. (82) and (84) will be compatible up to terms of order $(\mu_k-\mu)^2$ and $(\nu_k-\nu)^2$, which according to Eq. (10) should be neglected.

It is also possible to invert this transformation and, starting from p_k and q_k, to reconstruct μ_k, ν_k, ξ_k, and δ_k. Indeed, from the real and imaginary parts of p_k and from Eq. (78) we immediately obtain μ_k and ν_k. Then from the real part of q_k one finds ξ_k. Finally, given the imaginary part of q_k, knowing μ_k and ξ_k, we recover δ_k.

B. Inverse scattering method for the complex Toda chain

Here we use the fact that the well known Lax pair [29] for the standard Toda chain also holds for its complex version. Below we shall follow the notations and the approach in the monograph of Ref. [2].

The complex Toda chain can be written down as the compatibility condition for the two systems

$$F_{k+1}(\tau,\lambda)=U_k(\tau,\lambda)F_k(\tau,\lambda), \tag{91a}$$

$$U_k(\tau,\lambda)=\begin{pmatrix} p_k(\tau)+\lambda & e^{q_k(\tau)} \\ -e^{-q_k(\tau)} & 0 \end{pmatrix}, \tag{91b}$$

$$\frac{dF_k}{d\tau}=V_k(\tau,\lambda)F_k(\tau,\lambda), \tag{92a}$$

$$V_k(\tau,\lambda)=\begin{pmatrix} 0 & -e^{q_k(\tau)} \\ e^{-q_{k-1}(\tau)} & \lambda \end{pmatrix}. \tag{92b}$$

It is easy to check that this condition reads

$$\frac{dU_k}{d\tau}=V_{k+1}(\tau,\lambda)U_k(\tau,\lambda)-U_k(\tau,\lambda)V_k(\tau,\lambda) \tag{93}$$

and that Eq. (93) holds identically with respect to the spectral parameter λ provided that q_k and p_k solve the complex Toda chain system. In our case we have a Toda chain with finite number of nodes, equal to the number of solitons N. Therefore the analog for the scattering matrix is

$$T_N(\tau,\lambda)=U_N(\tau,\lambda)U_{N-1}(\tau,\lambda)\cdots U_2(\tau,\lambda)U_1(\tau,\lambda) =\begin{pmatrix} a_N^+(\tau,\lambda) & -b_N^-(\tau,\lambda) \\ b_N^+(\tau,\lambda) & a_N^-(\tau,\lambda) \end{pmatrix}. \tag{94}$$

The τ dependence of $T_N(\tau,\lambda)$ is determined by

$$\frac{dT_N}{d\tau}=V_{N+1}(\tau,\lambda)T_N(\tau,\lambda)-T_N(\tau,\lambda)V_1(\tau,\lambda). \tag{95}$$

Due to the fact that $E_{1,0}=E_{N+1,N}=0$, we have

$$V_{N+1}(\tau,\lambda)=\begin{pmatrix} 0 & 0 \\ e^{-q_N(\tau)} & \lambda \end{pmatrix}, \tag{96a}$$

$$V_1(\tau,\lambda)=\begin{pmatrix} 0 & e^{-q_1(\tau)} \\ 0 & \lambda \end{pmatrix} \tag{96b}$$

and, as a consequence, from Eq. (95) we find

$$\frac{da_N^+}{d\tau}=0. \tag{97}$$

Therefore, the (1,1) matrix element of $a_N^+(\tau,\lambda)$ is the generating functional of the integrals of motion of the complex Toda chain. From Eq. (94) we immediately find that a_N^+ is a polynomial of order N with respect to λ and therefore Eq. (97) provides us with N complex integrals of motion. According to Liouville's theorem these are enough for the integrability of the CTC, provided they are in involution. This fact will be proved in Sec. IV C. We will finish this section by presenting the recurrent procedure of calculating $T_{N;11}$ as functionals of p_k and q_k.

Indeed, it is easy to find that

$$a_0^+=1, \tag{98a}$$

$$a_1^+=p_1+\lambda, \tag{98b}$$

$$a_2^+=(p_1+\lambda)(p_2+\lambda)-e^{q_2-q_1}. \tag{98c}$$

Let us analyze the expression for a_2^+. It gives us two nontrivial integrals (the coefficients before λ^1 and λ^0)

$$I_1=p_1+p_2=\text{ const}, \tag{99a}$$

$$I_2=p_1p_2-e^{q_2-q_1}=\text{ const}. \tag{99b}$$

The first of these is the analog of the momentum conservation; in KS notations it corresponds to the conservation of μ and ν. For $N=2$ the second integral in Eq. (99), I_2, is related to Λ_2 used by Karpman and Solov'ev through

$$\Lambda_2=I_1^2-4I_2. \tag{100}$$

In addition a direct calculation allows us to conclude that the generic solutions of the KS and CTC with $N=2$ are equivalent.

The recurrent relation, which allows us to construct the integrals of motion of the CTC for any N, has the form

$$a_{N+1}^+(\tau,\lambda)=(p_{N+1}(\tau)+\lambda)a_N^+(\tau,\lambda)-e^{q_{N+1}(\tau)-q_N(\tau)}a_{N-1}^+(\tau,\lambda). \tag{101}$$

C. Involutivity of the integrals of motion

Here we use the standard classical r-matrix approach. Is well known [2], the Poisson brackets between the matrix elements of the Lax matrix $U_n(\lambda)$ can be written down in the compact form

$$\{U_n(\lambda)\underset{,}{\otimes}U_m(\mu)\}=[r(\lambda-\mu),U_n(\lambda)\otimes U_m(\mu)]\delta_{nm}, \tag{102}$$

where $r(\lambda-\mu)$ is the canonical classical r matrix given by

$$r(\lambda-\mu)=\frac{P}{\lambda-\mu}, \tag{103a}$$

$$P=\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}. \tag{103b}$$

On the left-hand side of Eq. (102) we have used the notation $\{X\underset{,}{\otimes}Y\}$,

$$\{X\underset{,}{\otimes}Y\}_{ij,kl}\equiv\{X_{ik},Y_{jl}\}, \tag{104}$$

which must be consistent with the definition of the tensor product: $(X\otimes Y)_{ik,jl}=X_{ik}Y_{jl}$.

It is a standard procedure to prove that from Eq. (102) there immediately follows that the scattering matrix $T(\lambda)$ satisfies (here and below we skip the index N)

$$\{T(\lambda)\underset{,}{\otimes}T(\mu)\}=[r(\lambda-\mu),T(\lambda)\otimes T(\mu)]. \tag{105}$$

These are 16 equations for the 16 matrix elements of the direct products. We shall list only several of them:

$$\{a^+(\lambda),a^+(\mu)\}=\{a^-(\lambda),a^-(\mu)\}=0, \tag{106a}$$

$$\{b^+(\lambda),b^+(\mu)\}=\{b^-(\lambda),b^-(\mu)\}=0, \tag{106b}$$

$$\{a^+(\lambda),a^-(\mu)\}=-\frac{b^+(\lambda)b^-(\mu)-b^-(\lambda)b^+(\mu)}{\lambda-\mu}, \tag{106c}$$

$$\{b^+(\lambda),b^-(\mu)\}=\frac{a^+(\mu)a^-(\lambda)-a^-(\mu)a^+(\lambda)}{\lambda-\mu}, \tag{106d}$$

$$\{b^+(\lambda),a^\pm(\mu)\}=\pm\frac{a^\pm(\lambda)b^+(\mu)-b^+(\lambda)a^\pm(\mu)}{\lambda-\mu}, \tag{106e}$$

$$\{b^-(\lambda),a^\pm(\mu)\}=\mp\frac{a^\pm(\lambda)b^-(\mu)-b^-(\lambda)a^\pm(\mu)}{\lambda-\mu}. \tag{106f}$$

In particular, from Eq. (106a), we find that $a^+(\lambda)$ and $a^+(\mu)$ are in involution for arbitrary values of the spectral parameters λ and μ. This immediately leads us to the conclusion that their coefficients I_n,

$$a^+(\lambda)=\lambda^N+\sum_{n=1}^{N}I_n\lambda^{N-n}, \tag{107a}$$

$$\{I_k,I_n\}=0, \tag{107b}$$

must be in involution for all values of k, $n=1,\dots,N$.

Therefore we see that the conditions of Liouville's theorem are fulfilled, and that the CTC is a completely integrable Hamiltonian system. In particular, from the recurrent relations (101) we derive

$$I_1^{(N)}=\sum_{i=1}^{N}p_i, \tag{108a}$$

$$I_2^{(N)}=\sum_{i<j}^{N}p_ip_j-\sum_{j=1}^{N-1}e^{q_{j+1}-q_j}=\frac{1}{2}(I_1^{(N)})^2-H_N, \tag{108b}$$

where

$$H_N=\frac{1}{2}\sum_{j=1}^{N} p_j^2+\sum_{j=1}^{N-1} e^{q_{j+1}-q_j}. \tag{109}$$

In the real case H_N becomes the Hamiltonian of the RTC.

D. Reduction of the Toda chain

Consider the Toda chain (87) with $N>2$. Our goal now is to reduce this system to an effective one for $N=2$. Let us write down equations for the differences $q_{k+1}-q_k$:

$$\frac{d^2(q_2-q_1)}{d\tau^2}=-2e^{q_2-q_1}, \tag{110}$$

for the case $N=2$, and

$$\frac{d^2(q_{k+1}-q_k)}{d\tau^2}=e^{q_{k+2}-q_{k+1}}-2e^{q_{k+1}-q_k}+e^{q_k-q_{k-1}} \tag{111}$$

for the case $N>2$.

Now we require that all differences (111) be expressed by only one effective difference $q_2^{\text{eff}}-q_1^{\text{eff}}$ in such a way that each of Eqs. (111) goes into Eq. (110) for the effective variables. We find that this is fulfilled if

$$\tilde{q}_{k+1}-\tilde{q}_k=q_2^{\text{eff}}-q_1^{\text{eff}}+\ln b_k, \tag{112a}$$

$$b_k=k(N-k), \quad k=1,\ldots,N-1. \tag{112b}$$

Only in this section variables with a tilde refer to the reduced system; they all are expressed in terms of just $q_2^{\text{eff}}-q_1^{\text{eff}}$. For readers acquainted with the theory of simple Lie algebras, we point out that the CTC (87) is related to the algebra $sl(N)$, while the one with $N=2$ is related to $sl(2)$. Therefore the reduction to an effective two-node CTC described above would correspond to the principal embedding of $sl(2)$ into $sl(N)$. This is always possible, because $sl(2)$ has an N-dimensional irreducible representation for any N.

We introduce the quantities $r^{(N)}$ and $\psi^{(N)}$ by the equations

$$\tilde{q}_{k+1}-\tilde{q}_k=-2\nu\left(r^{(N)}-\frac{1}{2\nu}\ln b_k\right)+\ln 4\nu^2 -i(\psi^{(N)}+\pi-2\mu r^{(N)}), \tag{113a}$$

$$r^{(N)}=\xi_2^{\text{eff}}-\xi_1^{\text{eff}}, \tag{113b}$$

$$\psi^{(N)}=\delta_2^{\text{eff}}-\delta_1^{\text{eff}}. \tag{113c}$$

Thus all distances and phase differences are expressed through only two effective parameters $r^{(N)}$ and $\psi^{(N)}$, and the N-node CTC is reduced to an effective two-node CTC for $q_2^{\text{eff}}-q_1^{\text{eff}}$.

Due to the reduction, the N integrals of motion will be expressed as functions of only $I_1^{(N)}$ and $I_2^{(N)}$, which in turn are naturally related to the integrals of the corresponding effective CTC. In order to make comparison with [12] simpler, we introduce

$$\tilde{\Lambda}_N^2\equiv 4\left(H_N-\frac{1}{2N}(I_1^{(N)})^2\right)=\frac{N(N^2-1)}{6}(\tilde{\Lambda}_2^{(N)})^2, \tag{114}$$

where $\tilde{\Lambda}_2^{(N)}$ and $\tilde{\Lambda}_N$ are the Karpman-Solov'ev integral and its generalization for the N-soliton reduced system.

Important cases in which the dynamics of the N-node CTC is effectively related to the specific choice of initial conditions (IC's) are

$$p_k(0)=-(\mu+i\nu), \tag{115a}$$

$$\xi_{k+1}(0)-\xi_k(0)=r_0, \tag{115b}$$

$$\delta_{k+1}(0)-\delta_k(0)=\delta_0, \tag{115c}$$

where μ, ν, r_0, and δ_0 are k-independent constants. Of course, the integrals of motion $I_k^{(N)}$ are all expressed in terms of these IC's. Imposing on Λ_N these IC's we find

$$\Lambda_N^2=(N-1)\Lambda_2^2, \tag{116}$$

where Λ_2^2 is again the Karpman-Solov'ev integral.

Note that these two special cases are compatible only for $N=3$. For $N>3$ the reduction requires that $\tilde{\xi}_{k+1}(0)-\tilde{\xi}_k(0)$ depends on k; see Eq. (112). Therefore strict results using both the reduction and the special initial conditions can be obtained only for $N=3$. For $N>3$ such considerations are approximate. We note that the maximal values of $\ln(b_k/b_{k-1})$ are less than 10% for $N\leqslant 7$ and $r_0=6$, and about 7% for $r_0=8$.

Introducing, as in [12], the notations $\Lambda_2=m_2+in_2$ and $\tilde{\Lambda}_2^{(N)}=\tilde{m}_N+i\tilde{n}_N$ and setting $\tilde{\Lambda}_N=\Lambda_N$ from Eqs. (114) and (116), we obtain

$$\tilde{m}_N=\left(\frac{6}{N(N+1)}\right)^{1/2}m_2, \tag{117a}$$

$$\tilde{n}_N=\left(\frac{6}{N(N+1)}\right)^{1/2}n_2. \tag{117b}$$

Next we can apply the Karpman-Solov'ev's analysis to these effective two-soliton systems and consider three different situations corresponding to the choice of the integral Λ_2. We formulate the results, which allow us to explain the data in Fig. 1.

(i) Let $m_N\neq 0$. Then the interaction is repulsive. In order to explain the data in Fig. 1 we consider only the asymptotics of the solution $R^{(N)}(t)=\tilde{\xi}_N(t)-\tilde{\xi}_1(t)$. Using the results in [12] we find that, for $t\to\pm\infty$,

$$R_N(t)\simeq\pm 2\left(\frac{6(N-1)^2}{N(N+1)}\right)^{1/2}m_2 t+\frac{N-1}{2\nu}\ln\frac{4\nu^2}{m_2^2+n_2^2} +\frac{N-1}{2\nu}\ln\frac{N(N+1)}{6}-\frac{1}{2\nu}\ln[(N-1)!]^2. \tag{118}$$

The last two summands balance each other to a high degree when $N>3$ because $6/N(N-1)$ is actually the mean value of all b_k, and for $N=3$ they cancel each other. From Eq.

(118) we can evaluate $\beta_N(t) = R_N(t)/R_N(0)$ for large values of t. In particular, for β_2 and β_3, we obtain

$$\beta_2(t) \simeq \frac{2m_2 t + \frac{1}{2\nu}\ln\frac{4\nu^2}{m_2^2+n_2^2}}{R_2(0)}, \tag{119a}$$

$$\beta_3(t) \simeq \frac{\sqrt{2}m_2 t + \frac{1}{2\nu}\ln\frac{4\nu^2}{m_2^2+n_2^2}}{R_2(0)}. \tag{119b}$$

Consequently $\beta_3 < \beta_2$, and the numerical results on Fig. 1 are described very well.

The time dependence of β_N can also be derived in the generic case; see [39]. For example, if the IC is provided by Eq. (115) we find [39]

$$\beta_N(t) \simeq \frac{16\nu e^{-\nu r_0}}{(N-1)r_0}\cos\frac{\pi}{N+1}\sin\frac{\delta_0}{2}t + 1 - \frac{1}{(N-1)\nu r_0}\ln(2^{N-1}\gamma_N),$$

$$\gamma_N = \prod_{k=1}^{N-1}(\cos\theta_j - \cos\theta_N), \tag{120}$$

$$\gamma_1 = 1,$$

$$\theta_k = \frac{k\pi}{N+1}.$$

These formulas are compatible with the BPM for $\delta_0 \simeq \pi$. The interval of validity grows with r_0. In general the range of validity of these formulas is related to the problem of finding the class of initial conditions, for which CTC is an adequate approximation; see also Sec. IV F below.

(ii) Let now $m_N = 0$, $n_N \neq 0$. Then in the N-soliton case we may have also a periodic solution; see formulas (3.31)–(3.35) in [12]. For equal amplitudes the parameter α_1 in this formula vanishes and the solution becomes singular. Using the reduction we find, that such singularity takes place also for the three-soliton case with equal amplitudes and $\delta_0 = 0$.

For the three-soliton case we also find explicit expression for the period

$$T_3 = \frac{\pi}{2\nu|\tilde{n}_3|} = \frac{\sqrt{2}\pi}{2\nu|\tilde{n}_2|} = \sqrt{2}T_2, \tag{121}$$

i.e., the period lengthens with the growth of N.

(iii) If $m_N = 0$, $n_N = 0$, and $N = 3$ then there is no essential difference from the two-soliton case due to the reduction described above.

E. CTC versus RTC

We already mentioned above that soliton interaction in an infinitely long soliton chain has been conjectured to be described by the infinite real Toda chain [27,26]. In these papers only the soliton positions ξ_k are taken into account and in addition it was assumed that the phase differences $\phi_{0;kn}$ remain constant in time. As we can see, this last assumption is not compatible with the system (35)–(38) derived above.

Let us now briefly discuss the relations between these facts and the CTC. If we introduce the real and the imaginary parts of q_k by $q_k = P_k + i\Psi_k$ then the CTC will be rewritten as a system of $2N$ equations for the $2N$ real variables P_k and Ψ_k. If we now impose the condition $\Psi_{k+1} - \Psi_k = \Psi$ $=$const, we then obtain

$$\frac{d^2P_k}{d\tau^2} = \cos\Psi(e^{P_{k+1}-P_k} - e^{P_k-P_{k-1}}), \tag{122}$$

$$\frac{d^2\Psi_k}{d\tau^2} = \sin\Psi(e^{P_{k+1}-P_k} - e^{P_k-P_{k-1}}). \tag{123}$$

This system is consistent only provided $\sin\Psi = 0$, i.e., $\Psi = 0$ or π, as has been chosen in [31]. These two cases are substantially different. Indeed, for $\Psi = \pi$ we obtain the RTC in its standard form, solved by [29]. In fact we used this solution in deriving Eq. (120). For $\Psi = 0$ we obtain a different, and much less studied, version of the RTC, one with a ''wrong'' sign on the right-hand side, which has singular solutions; we will call it the singular RTC (SRTC). This reflects the fact that the CTC has a class of solutions that become singular for finite values of t; such singularity exists also for $N = 2$; see (ii) in Sec. IV D above.

Recently it was also checked numerically that the SRTC gives an adequate description of the soliton positions for the case when we have a train of three solitons with initial conditions: $\nu_k = \frac{1}{2}$, $\mu_k = 0$, $r_0 = 8$, and $\delta_0 = 0$ or π; see [31] for values of t up to about 50. Our more recent check shows, that SRTC gives a fairly good description of the soliton interactions excluding the neighborhood of its singular points; see Fig. 4 below.

Another important difference between the RTC and the CTC is that the CTC has a much larger class of asymptotic states than the RTC. The only possible asymptotics of the RTC can be described as ''free solitons,'' each one moving with its own velocity [29]. At the same time the CTC also possesses asymptotic states, where some (or all) of the solitons form bound states; see, e.g., Figs. 3(d)–3(g).

The CTC is derived in a natural way from the generalized KS equations. We view the N-node CTC as a natural generalization of the results of the N-node RTC [31]. The generic N-soliton solution of the NLS equation can be viewed as a dynamical system with $2N$ degrees of freedom. The same holds true also for the GKS and for a CTC with N nodes. From this point of view the CTC is more adequate for describing the soliton interactions, and may be expected to do so for wider a class of IC's than the RTC.

Formally the solutions of the CTC can be obtained from those of the RTC by making the corresponding coefficients complex. As a result we may expect a richer variety of solutions and asymptotic behaviors. From general considerations it follows that the asymptotics of the RTC are described by ''free'' solitons, each one moving with its own asymptotic velocity $\hat{\mu}_k$, such that $\hat{\mu}_k \neq \hat{\mu}_n$ for $k \neq n$. In the case of the CTC the asymptotic is described by the ''complex velocities'' $\hat{\mu}_k + i\hat{\nu}_k$, where $\hat{\mu}_k$ characterizes the veloc-

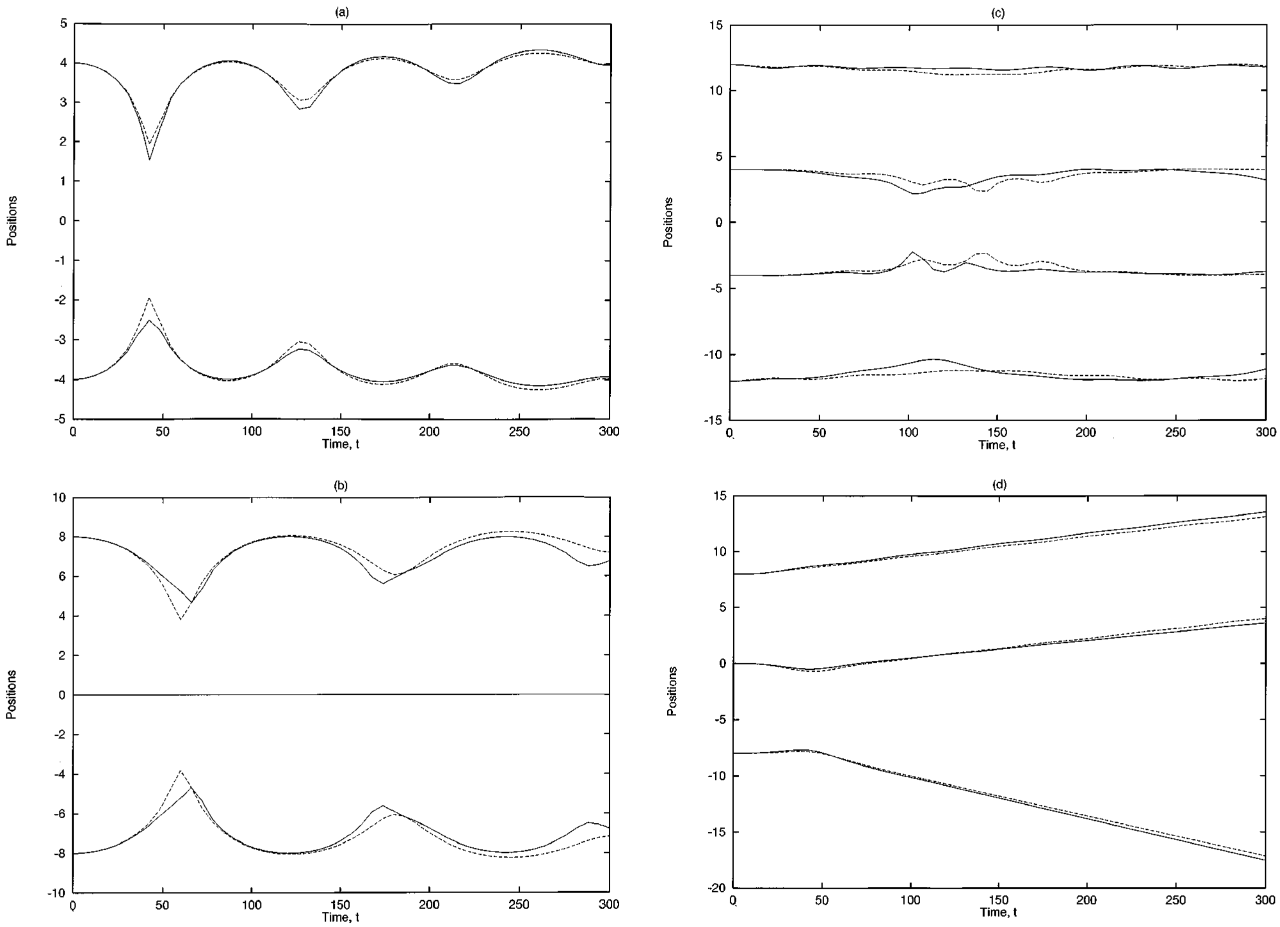

FIG. 3. Comparison between the CTC and BPM for other initial conditions. (a) The positions of a two-soliton interaction with initial state $\mu_k(0)=0$; $\nu_k(0)=\frac{1}{2}$; $r_0=8$; $k=1$ and 2; and $\delta_2(0)-\delta_1(0)=-0.05\pi$. Solid line, BPM, dashed line, CTC. (b) The positions of a three-soliton interaction with initial state $\mu_k(0)=0$; $\nu_k(0)=\frac{1}{2}$; $r_0=8$; $k=1$, 2, and 3; $\delta_1(0)=\delta_3(0)=0$; and $\delta_2(0)=-0.05\pi$. Solid line, BPM; dashed line, CTC. (c) The positions of a four-soliton interaction with initial states $\mu_k(0)=0$, $\nu_1(0)=\nu_3(0)=0.95/2$, $\nu_2(0)=\nu_4(0)=1.05/2$, $r_0=8$, and $\delta_k(0)=0$. $k=1$, 2, 3, and 4. Solid line, BPM; dashed line CTC. (d) The positions of a three-soliton interaction with initial states $\mu_k(0)=0$, $\nu_1(0)=0.95/2$, $\nu_2(0)=\frac{1}{2}$, $\nu_3(0)=1.05/2$, $r_0=8$, $\delta_1(0)=\delta_3(0)=0$, and $\delta_2(0)=\pi/2$. $k=1$, 2, and 3. Solid line, BPM; dashed line CTC. (e) The velocities of a three-soliton interaction with initial states $\mu_k(0)=0$, $\nu_1(0)=0.95/2$, $\nu_2(0)=\frac{1}{2}$, $\nu_3(0)=1.05/2$, $r_0=8$, $\delta_1(0)=\delta_3(0)=0$, and $\delta_2(0)=\pi/2$. $k=1$, 2, and 3; CTC. (f) The velocities of a three-soliton interaction with initial states $\mu_k(0)=0$, $\nu_1(0)=0.95/2$, $\nu_2(0)=1/2$, $\nu_3(0)=1.05/2$, $r_0=8$, $\delta_1(0)=\delta_3(0)=0$, and $\delta_2(0)=\pi/2$. $k=1$, 2, and 3; BPM. (g) The amplitudes of a three-soliton interaction with initial states $\mu_k(0)=0$, $\nu_1(0)=0.95/2$, $\nu_2(0)=1/2$, $\nu_3(0)=1.05/2$, $r_0=8$, $\delta_1(0)=\delta_3(0)=0$, and $\delta_2(0)=\pi/2$. $k=1$, 2, and 3; BPM.

ity and $\hat{\nu}_k$ the amplitude of the kth soliton. Again we may have $\hat{\mu}_k+i\hat{\nu}_k\neq\hat{\mu}_n+i\hat{\nu}_n$ for $k\neq n$, but now this does not necessarily mean that $\hat{\mu}_k\neq\hat{\mu}_n$. This explains why the asymptotics of the solutions of the CTC are richer than those of the RTC. In particular, it is known that the CTC has singular solutions that ''blow up'' for finite values of t; see [32]. We show examples of such solutions for $N=2$ and on Figs. 4; note that these solutions are both singular and periodic.

In addition, we see that the asymptotics of the CTC may include bound states of two (or more) solitons which stay equidistant with a very good precision; their velocities slightly oscillate around a common average value, see Figs. 3(e)–3(g) below. Of course this is possible only for solitons with different amplitudes.

This is also consistent with what is already known from the exact results for two-soliton interactions. For $N=2$ the CTC is equivalent to the KS system whose analytic solution was obtained in [12]. These results show that for certain choices of the initial conditions we find periodic and singular solutions.

F. GKS and CTC: Domain of validity

An important problem in this context is to describe more precisely the domain of validity of both the CTC and GKS models. In order to show that this domain is larger than a small neighborhood of the initial conditions (115) we present several examples, illustrating that CTC may fairly well describe the soliton interactions under a large variety of IC's. An analysis of the GKS has been performed in [34], where it was shown that the GKS reasonably describes the interaction of large sequences of unequal soliton pulses. A comparison between the GKS and the CTC was given above in Sec. IV A.

In all cases below we choose the solitons to be equidistant with $r_0=6$ and 8 and with vanishing initial velocities. For

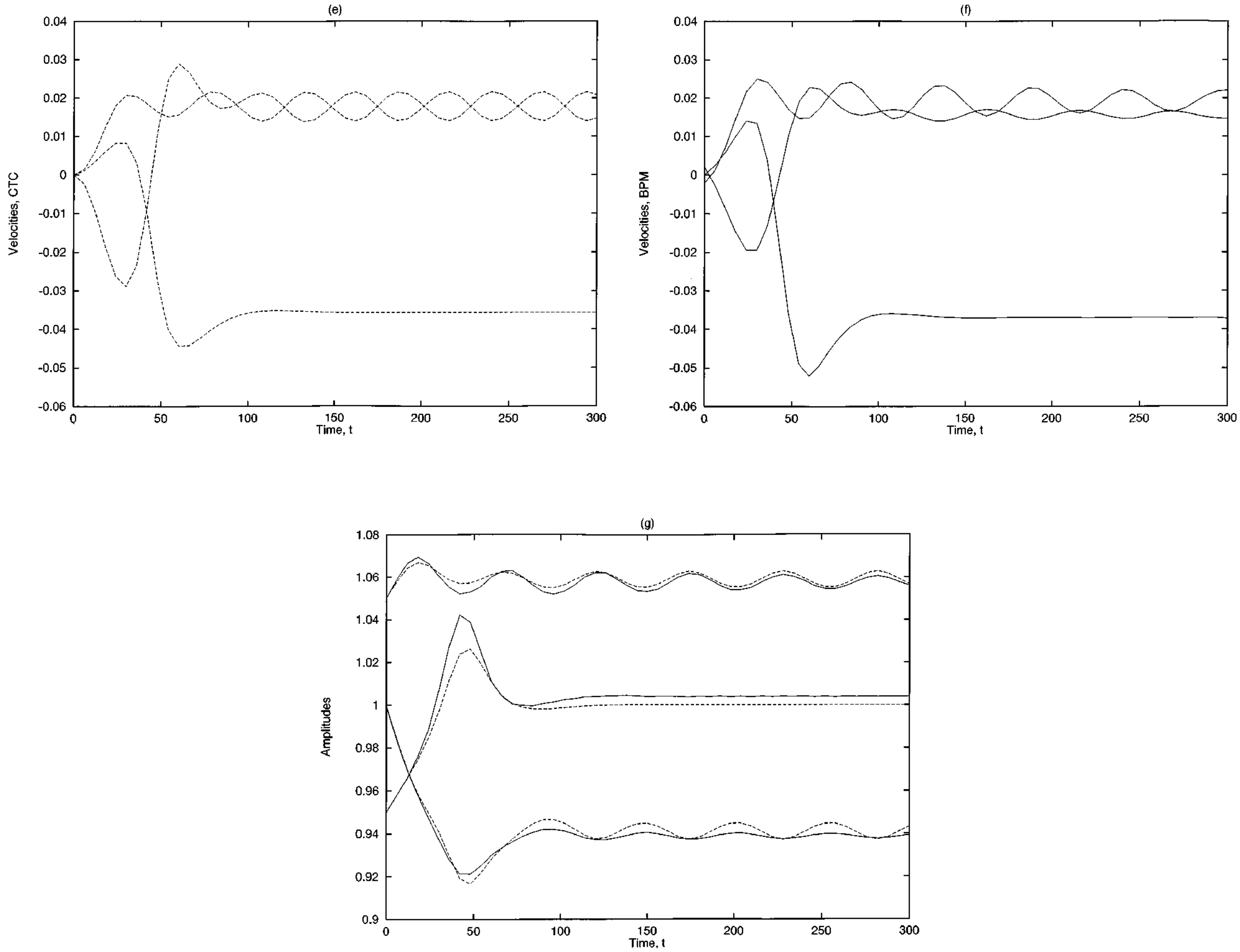

FIG. 3 (*Continued*).

brevity and clarity let us introduce the following notations for the sets of initial amplitudes and phases:

$$A_0^{(N)} \equiv \{2\nu_k(0)=1.0, k=1,\ldots,N\},$$

$$A_1^{(2)} \equiv \{2\nu_1(0)=0.95, 2\nu_2(0)=1.0\},$$

$$A_2^{(2)} \equiv \{2\nu_1(0)=1.0, 2\nu_2(0)=1.05\},$$

$$A_1^{(3)} \equiv \{2\nu_1(0)=0.95, 2\nu_2(0)=1.0, 2\nu_3(0)=1.05\},$$

$$A_2^{(4)} \equiv \{2\nu_1(0)=2\nu_3(0)=0.95, 2\nu_2(0)=2\nu_4(0)=1.05\},$$

$$D_0^{(N)} \equiv \{\delta_{k+1}(0)-\delta_k(0)=0, k=1,\ldots,N-1\},$$

$$D_1^{(N)} \equiv \{\delta_{k+1}(0)-\delta_k(0)=\pi, k=1,\ldots,N-1\},$$

$$D_2^{(2)} \equiv \{\delta_1(0)=0, \delta_2(0)=-0.05\pi\},$$

$$D_2^{(3)} \equiv \{\delta_1(0)=0, \delta_2(0)=-0.05\pi, \delta_3(0)=0\},$$

$$D_3^{(3)} \equiv \{\delta_1(0)=0, \delta_2(0)=\pi/2, \delta_3(0)=\pi\},$$

$$D_4^{(3)} \equiv \{\delta_1(0)=0, \delta_2(0)=\pi/2, \delta_3(0)=0\},$$

$$D_3^{(2)} \equiv \{\delta_1(0)=0, \delta_2(0)=\pi/2\},$$

$$D_4^{(2)} \equiv \{\delta_1(0)=\pi/2, \delta_2(0)=0\}.$$

In Figs. 3(a) and 3(b), we see very good quantitative agreement between the CTC and the BPM for two- and three-soliton interactions, with IC's given by $A_0^{(2)}, D_2^{(2)}$ and $A_0^{(3)}, D_2^{(3)}$, respectively. Here and below we assume $r_0=8$ and $\mu_k(0)=0$. The choice of this initial phase difference avoids the singularity of the corresponding CTC solution.

On Fig. 3(c) we show four solitons with $A_2^{(4)}, D_0^{(4)}$ like in [23]; we again find a good agreement with the BPM for values of t up to 120 and above 170.

Very interesting is the situation depicted in Fig. 3(d), where we consider three solitons with $A_1^{(3)}, D_4^{(3)}$. Here we find that two of the solitons form an asymptotic bound state, while the third one separates off. This is confirmed also by Figs. 3(e) and 3(f), where we show the velocities of the solitons evaluated with the CTC [Fig. 3(e)] and BPM [Fig. 3(f)], respectively. Here CTC gives a good agreement with the BPM, which also extends to the description of the soliton amplitudes; see Fig. 3(g).

This example also illustrates the fact that a knowledge of only two-soliton interactions cannot be sufficient for descriptions of the three- and N-soliton ones. Indeed, we know that

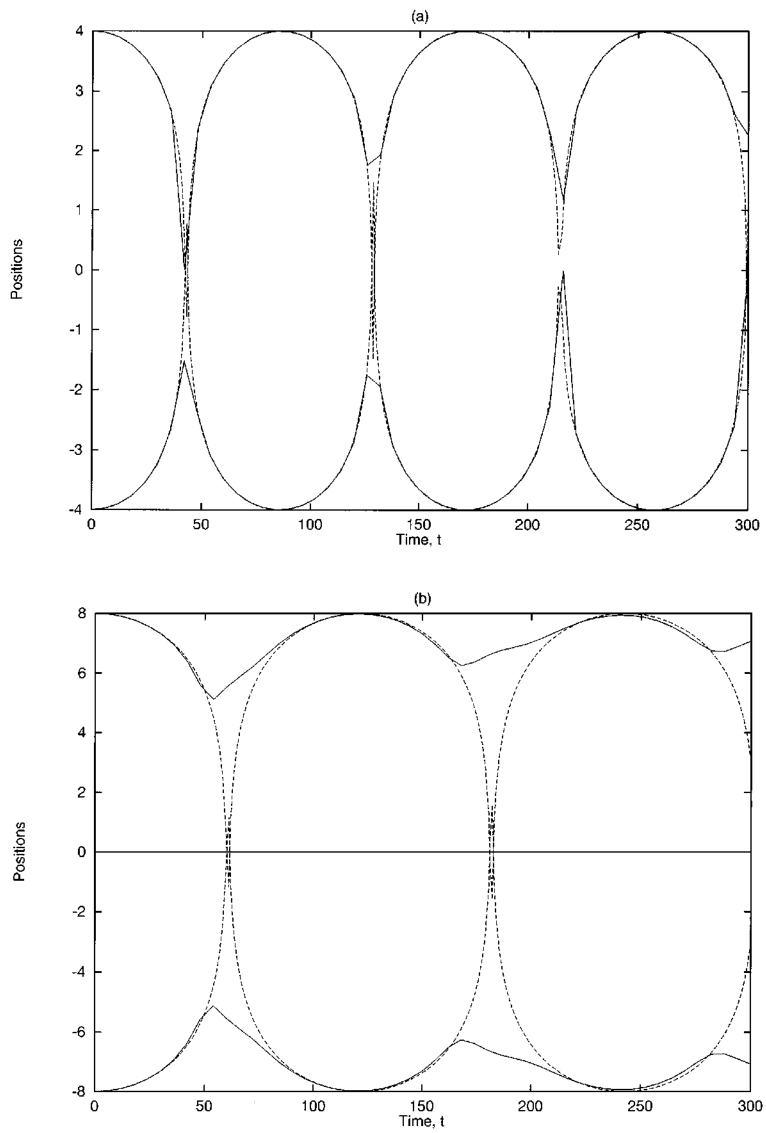

FIG. 4. Singular solutions of the CTC for $N=2$ and 3, and a comparison with BPM. (a) The positions of a two-soliton interaction with $\nu_1(0)=\nu_2(0)=\frac{1}{2}$, $\mu_1(0)=\mu_2(0)=0$, $\delta_1(0)-\delta_2(0)=0$, and $r_0=8$. Solid lines, BPM; dashed lines, CTC. (b) The positions of a three-soliton interaction with $\nu_k(0)=\frac{1}{2}$; $\mu_k(0)=0$, $\delta_k(0)=0$, $k=1$, 2, and 3; and $r_0=8$. Solid lines, BPM; dashed lines, CTC.

two solitons can form a bound state only if their initial velocities are equal, the phase difference is equal to zero and the amplitudes are different; for example, $A_1^{(2)}, D_0^{(2)}$.

At the same time, in the case of Figs. 3(d)–3(g), the first and second solitons are characterized by $A_1^{(2)}, D_3^{(2)}$ and the second and the third solitons by $A_2^{(2)}, D_4^{(2)}$. Both these configurations in the two-soliton case lead to a repulsive interaction, so if we apply the two-soliton intuition for the three-soliton case, we should expect that the three solitons separate off. However, what we see in Figs. 3(d)–3(g) is qualitatively different, and can be predicted in no case by the two-soliton solution.

In Figs. 4(a) and 4(b) we show the two- and three-soliton interaction with $A_0^{(2)}, D_0^{(2)}$ and $A_0^{(3)}, D_0^{(3)}$, respectively. Again we see that the CTC qualitatively describes the soliton interactions very well, with the exception of the regions where the singularities occur. Another similar example for three-soliton interactions can be seen in Fig. 4(b). We see that the corresponding BPM solutions also tend to show a periodic behavior, and that the periods of both solutions are roughly the same. Finally, the ratio of the periods for $N=2$ and 3 are related through formula (121).

Up to here we analyzed the soliton interactions of the pure NLS with a vanishing right-hand side. Applying the same kind of approximations to the perturbed versions of the system (35)–(38), i.e., with $iR[u]\neq 0$, we obtain perturbed versions of the CTC.

However, in doing this consistently, one may meet difficulties. First of all, note that in both $N_k^{(p)}$ and $M_k^{(p)}$ one obtains additional self-interacting terms, and now one should perform a balance between them and the already present exponentially small terms $S_{k,n}$ and $C_{k,n}$. If the latter are the dominating terms, due, say to the fact that $N_k^{(p)}$ and $M_k^{(p)}$ are either 0 or are multiplied by a small constant, then we indeed obtain a perturbed version of the CTC. If, however, the perturbative terms $N_k^{(p)}$ and $M_k^{(p)}$, $p>0$ happen to be the leading ones, then no CTC may result; this situation must be considered separately.

Another case, to be considered separately is the case of solitons with unequal amplitudes. The derivation of the CTC in this case, strictly speaking, should be done separately. Indeed, if there is a substantial difference between ν_k and $\nu_{k\pm 1}$ we cannot replace ν_n in the arguments for the exponents $\exp(-|\beta_{k,k-1}|)$ and $\exp(-|\beta_{k,k+1}|)$ by the average amplitude ν. Then generically $\exp(-|\beta_{k,k-1}|)$ and $\exp(-|\beta_{k,k+1}|)$ will be of different orders of magnitude.

There is, however, a particular situation, namely, $\nu_1=\nu_3=\cdots$ and $\nu_2=\nu_4=\cdots$, when both terms on the right hand side of Eqs. (35) and (36) will be of the same order of magnitude, which, however, will be different for odd and even values of k. It is known that soliton trains of four in-phase unequal amplitude solitons demonstrate stable propagation [23,34]; see also Fig. 3(c).

Finally, let us briefly discuss the cases when the systems derived above fail to describe the soliton interaction. Of course, this may happen when the initial approximations (10) are destroyed by the evolution. After that neither the CTC, GKS, nor system (35)–(38) can be expected to work.

As an example of such a situation we point out the case of soliton collisions. It is well known that in the case of $A_0^{(N)}, D_0^{(N)}$, $N=2$ and 3, they attract each other, and as a result the distance between them diminishes and approximations (10) may fail. Somewhat unexpectedly we find that the CTC describes very well the soliton positions even for such IC's, failing to do so only in comparatively small regions around its singular points; see Fig. 4.

Let us briefly comment on the effects of perturbations on GKS and CTC. Of course this is an open field which requires more than one paragraph or paper to be covered.

Some of the perturbations (e.g., the bandwidth-limited amplification and nonlinear gain) do not violate the symmetry of the soliton train. Others (such as third-order dispersion) act in an unsymmetrical way on the each of the pulses. As a result the amplitudes of the pulses change substantially due to the perturbation and violate the initial approximations. The applicability of the Karpman-Solov'ev approach for the two-soliton case is investigated in detail in [38].

Other perturbations (such as TOD) may lead to substantial radiation, and thus the adiabatic approximation becomes invalid. Therefore, such processes cannot be investigated with the method proposed above. An analytical approach to the study of the effects of radiation was proposed recently in [40]. These problems are out of the scope of the present work.

In some cases (e.g., for bandwidth-limited amplification and nonlinear gain) the perturbed NLSE acquires the form of the Ginzburg-Landau equation. Then even for comparatively small values of the perturbation coefficients the NLS soliton pulses due to the interaction may transform into the exact solutions of the Ginzburg-Landau equation [36]. They differ from the soliton pulses by their amplitudes, widths, and most importantly, by their chirp. Their interaction should be described by other methods [21,36].

Although the list of these situations can probably be extended, we would like to stress again, that there definitely exist a manifold of initial data for the solitonlike pulses, for which system (35)–(38) and (or) the CTC give a proper and adequate description. It is known, that the asymptotics of the solutions of the standard RTC are given by the free motion of the particles [29], in our case solitons. Its generalization, the CTC, possesses a much richer variety of asymptotic behaviors, see Figs. 3 and 4. What would be ideal for the optical communications lines using solitons would be to achieve a stable propagation of soliton trains in which the solitons have nearly the same asymptotic velocities. Therefore, we believe that further studies of the CTC may help to shed more light on this problem.

V. CONCLUSIONS

We have generalized the Karpman-Solov'ev approach for an arbitrary number of $N>2$ interacting solitons. We have been also able to account for a large class of physically important perturbations. As a result we obtain a dynamical system of $4N$ equations describing the evolution of the soliton parameters under these perturbations.

As it should be expected, the right hand sides of the dynamical system of equations for the soliton parameters contain complicated nonlinear nearest-neighbor interaction terms. This system does not allow the superposition principle, so that the N-soliton interaction can not be reduced to separately interacting soliton pairs.

We have also proved that the dynamics of the train of N-soliton-like equidistant pulses with (nearly) equal amplitudes and velocities and moving according to the unperturbed NLS equation after some simplification reduces to the N-node complex Toda chain. If we choose the phase differences between the neighboring solitons to be π, we derive the RTC for their positions, which was analyzed earlier in [31].

It is possible also for some types of perturbations $iR[u]$ to derive the corresponding perturbed versions of the CTC. This fact can be used for describing the class of initial conditions under which the propagation of solitons will be stable, and compare them with the already known stable combinations, see [22,23].

Other important question outlined above are as follows: (a) To describe the class of initial data, for which the CTC adequately describes the soliton interactions. The examples given above show that it is larger than just a small region around the initial conditions (115) with $\delta_0 \simeq \pi$. (b) To use the explicit solutions of the CTC and their large-t asymptotics for a description of the behavior of the soliton trains. Work in these directions is under progress; see [39].

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APPENDIX A

Here we briefly describe the evaluation of the different types of integrals that appear in the calculations, and more specifically their asymptotic behavior for $a,b \to 0$ and $\beta \gg 1$. The simplest types of integrals that appear have the form

$$\mathcal{J}_p(a) = \int_{-\infty}^{\infty} \frac{dz\, e^{iaz}}{2\, \cosh^p z}, \tag{A1a}$$

$$J_p(a)=\int_{-\infty}^{\infty}\frac{dz\,e^{iaz}\sinh z}{2\,\cosh^p z}. \tag{A1b}$$

Using integration by parts, these integrals can be expressed by $\mathcal{J}_1(a)$ and $\mathcal{J}_2(a)$, which in turn are known from the tables of integrals [41]. We have

$$\mathcal{J}_p(a)=\frac{a^2+(p-2)^2}{(p-1)(p-2)}\mathcal{J}_{p-2}(a), \tag{A2a}$$

$$J_p(a)=\frac{ia}{p-1}\mathcal{J}_{p-1}(a) \tag{A2b}$$

and

$$\mathcal{J}_1(a)=\frac{\pi}{2\cosh\dfrac{a\pi}{2}}, \tag{A3a}$$

$$\mathcal{J}_2(a)=\frac{\pi a}{2\sinh\dfrac{a\pi}{2}}. \tag{A3b}$$

Obviously $\mathcal{J}_p(a)$ are even functions of a, while $J_p'(a)$ are odd functions of a. In addition, $\mathcal{J}_p(a)$ are Schwartz-type functions of a on the whole axis. In particular,

$$\mathcal{J}_{2s+1}(0)=\frac{(2s-1)!!}{(2s)!!}\,\frac{\pi}{2}, \tag{A4a}$$

$$\mathcal{J}_{2s}(0)=\frac{(2s-2)!!}{(2s-1)!!}, \tag{A4b}$$

and their first-order derivatives with respect to a vanish for $a\to 0$.

Let us now analyze the integrals $\mathcal{P}_p(a,b,\beta)$ and $\mathcal{Q}_p(a,b,\beta)$. We do not know exact explicit expressions for them in terms of a, b, and β. However from Eqs. (22) and (23) one concludes that they are Schwartz-type functions of their parameters in the strip $|\mathrm{Im}a|<p+1+b$. In our consideration we assume $p>1$ and $|\mathrm{Im}a|\ll 1$, $b\ll 1$, so a and $a_\pm=a\pm ib$ are always inside the above-mentioned strip of the complex plane. This fact allows us to assume that the derivatives of all orders of our functions will be smooth and bounded.

In the case when $|\beta|\gg 1$, we obtain

$$\frac{e^{iaz}}{\cosh[(1+b)z+\beta]}\simeq 2e^{ia_\pm z}e^{-|\beta|}(\cosh z-s_\beta\sinh z)\times[1+O(e^{-2|\beta|})], \tag{A5}$$

where $s_\beta=\mathrm{sgn}\beta$. After simple calculations using Eq. (A2), we obtain

$$\mathcal{P}_p(a,b,\beta)\simeq 4e^{-|\beta|}\left(1-\frac{is_\beta a_\pm}{p-1}\right)\mathcal{J}_{p-1}(a_\pm)\times[1+O(e^{-2|\beta|})], \tag{A6}$$

$$\mathcal{Q}_p(a,b,\beta)\simeq\frac{4e^{-|\beta|}}{(p-1)(p-2)}\mathcal{A}_p(a_\pm,\beta)\mathcal{J}_{p-2}(a_\pm)\times[1+O(e^{-2|\beta|})], \tag{A7}$$

$$\mathcal{A}_p(a_\pm,\beta)=s_\beta a_\pm^2+i(p-1)a_\pm-s_\beta(p-2). \tag{A8}$$

From these formulas we can easily also evaluate the behavior of the integrals

$$\mathcal{R}_p(a,b,\beta)=\int_{-\infty}^{\infty}\frac{dz\,e^{iaz}z}{\cosh^p z\,\cosh[(1+b)z+\beta]}=\frac{1}{i}\frac{d}{da}\mathcal{P}_p(a,b,\beta), \tag{A9}$$

$$\mathcal{S}_p(a,b,\beta)=\int_{-\infty}^{\infty}\frac{dz\,e^{iaz}z\sinh z}{\cosh^p z\,\cosh[(1+b)z+\beta]}=\frac{1}{i}\frac{d}{da}\mathcal{Q}_p(a,b,\beta), \tag{A10}$$

which appear in evaluating $\Xi_k^{(0)}[u]$ and $D_k^{(0)}[u]$. Further, when evaluating $\Xi_k^{(p)}[u]$ and $D_k^{(p)}[u]$ with $s\geq 1$, we also encounter integrals of the types

$$\mathcal{U}_{p,2}(a,b,\beta)=\int_{-\infty}^{\infty}\frac{dz\,e^{iaz}\sinh[(1+b)z+\beta]}{\cosh^p z\,\cosh^2[(1+b)z+\beta]}, \tag{A11}$$

$$\mathcal{W}_{p,2}(a,b,\beta)==\int_{-\infty}^{\infty}\frac{dz\,e^{iaz}\sinh z\,\sinh[(1+b)z+\beta]}{\cosh^p z\,\cosh^2[(1+b)z+\beta]}, \tag{A12}$$

which, after integration by parts are expressed through $\mathcal{P}_p$ and $\mathcal{Q}_p$,

$$\mathcal{U}_{p,2}(a,b,\beta)=\frac{\nu_k}{\nu_n}[ia\mathcal{P}_p(a,b,\beta)-p\,\mathcal{Q}_{p+1}(a,b,\beta)], \tag{A13}$$

$$\mathcal{W}_{p,2}(a,b,\beta)=\frac{\nu_k}{\nu_n}[ia\,\mathcal{Q}_p(a,b,\beta)-(p-1)\mathcal{P}_{p-1}(a,b,\beta)+p\mathcal{P}_{p+1}(a,b,\beta)]. \tag{A14}$$

These formulas are enough to evaluate the necessary expressions for the coefficients $N_k^{(p)}[u]$, $\dots$, $D_k^{(p)}[u]$.

APPENDIX B

Let us show now in more detail what kind of approximations we perform in deriving the generalized Karpman-Solov'ev system and the CTC. First we shall explain why the terms $\widetilde{R}_k^{(0)}[u]$ in Eq. (12) can really be neglected. Indeed, inserting the term $u_ku_m^*u_n$ into the right-hand sides of Eqs. (15)–(18) we obtain integrands with denominators of the type

$$\frac{1}{\cosh^p z_k \cosh z_m \cosh z_n}$$

$$\simeq e^{-|\beta_{kn}|-|\beta_{km}|}\frac{(\cosh z_k - s_{kn}\sinh z_k)(\cosh z_k - s_{km}\sinh z_k)}{\cosh^p z_k}$$
$$\times[1+O(e^{-2|\beta_{kn}|})+O(e^{-2|\beta_{km}|})]$$
$$\simeq O(e^{-2|\beta_{km}|}), \tag{B1}$$

where we have made use of Eq. (A5). Evaluated at $t=0$ the right-hand side of Eq. (B1) for the case $n=k+1$ and $m=k-1$ has an order ϵ^2, which means that they can be neglected. The other possible cases, when $|k-m|>1$ or $|k-n|>1$, lead to estimations of still higher order in ϵ.

Let us now briefly go into some detail about the derivation of the CTC, particularly about to the connection between the old variables ν_k, μ_k, ξ_k, and δ_k and the new ones q_k and p_k. We will evaluate the derivative of q_k. We start with the derivative of δ:

$$\frac{d\delta}{dt}=\frac{d}{dt}\left(\frac{1}{N}\sum_{p=1}^{N}\delta_p\right)$$

$$=2\mu^2+2\nu^2+\frac{2}{N}\sum_{p=1}^{N}[(\mu-\mu_p)^2+(\nu-\nu_p)^2]$$
$$\simeq 2\mu^2+2\nu^2, \tag{B2}$$

where we have neglected terms of the order $(\mu-\mu_p)^2$ and $(\nu-\nu_p)^2$.

Now we evaluate the derivative of q_k using Eq. (B2):

$$\frac{dq_k}{dt}=\frac{d}{dt}[-2\nu\xi_k+k\ \ln 4\nu^2-i(\delta_k+\delta+k\pi-2\mu\xi_k)]$$
$$=-4\nu\mu_k-i(2\mu_k^2+2\nu_k^2+2\mu^2+2\nu^2-4\mu\mu_k)$$
$$=-4\nu\mu_k-4i\nu\nu_k-2i[(\mu-\mu_k)^2+(\nu-\nu_k)^2]$$
$$\simeq -4\nu(\mu_k+i\nu_k)=-4\nu_k. \tag{B3}$$

If we now use the variables p_k and τ, we conclude that

$$\frac{dq_k}{d\tau}=p_k. \tag{B4}$$

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